# APP02 

# RETRIEVING THE FLEXURAL RIGIDITY OF A BEAM FROM DEFLECTION MEASUREMENTS 

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#### Abstract

A rigorous investigation into the identification of the heterogeneous flexural rigidity coefficient from deflection measurements recorded along a beam in the presence of a prescribed load is presented. Mathematically, the problem reduces to the need to solve the Euler-Bernoulli steady-state beam equation subject to appropriate boundary conditions. Conditions for the uniqueness and continuous dependence on the input data of the solution of the inverse problem for simply supported beams are established and, in particular, it is shown that the operator which maps an input deflection into an output flexural rigidity is Holder continuous. Since the inverse problem can be recast in the form of a Fredholm integral equation of the first kind, numerical results obtained using various methods, such as Tikhonov's regularization, singular value decomposition and mollification are discussed.


## NOMENCLATURE

$E$ Modulus of elasticity
I Moment of inertia
$L$ Length of the beam
$M$ Bending moment
a Flexural rigidity
$f$ Transversely distributed load
$p$ Amount of noise
$u$ Transverse deflection
$\alpha$ Inverse of the flexural rigidity
$\lambda$ Regularization parameter
$\sigma$ Standard deviation

## INTRODUCTION

In the Euler-Bernoulli beam theory, it is assumed that the plane cross-sections perpendicular to the axis of the beam remain plane and perpendicular to the axis after deformation, resulting in the transverse deflection $u$ of the beam being governed by the fourth-order ordinary differential equation

$$
\begin{equation*}
\frac{d^{2}}{d x^{2}}\left(a(x) \frac{d^{2} u}{d x^{2}}(x)\right)=f(x), \quad 0<x<L \tag{1}
\end{equation*}
$$

where $L$ is the length of the beam, $f$ is the transversely distributed load and the spacewise dependent conductivity $a=E I$ (often called flexural rigidity) is the product of the modulus of elasticity $E$ and the moment of inertia $I$ of the cross section of the beam about an axis through its centroid at right angles to the cross section. It should be noted that eqn.(1) can be split into a system of two secondorder ordinary differential equations, namely,

$$
\begin{equation*}
M(x)=a(x) \frac{d^{2} u}{d x^{2}}(x), \quad 0<x<L \tag{2}
\end{equation*}
$$

$$
\begin{equation*}
\frac{d^{2} M}{d x^{2}}=f(x), \quad 0<x<L \tag{3}
\end{equation*}
$$

where $M$ is the bending moment. Using a simple finite element methodology, it can be seen that the primary variables
associated with eqn.(1) are the deflection $u$ and the slope $d u / d x$, whilst the bending moment $M=a\left(d^{2} u / d x^{2}\right)$ and the shear force $(d / d x)\left(a d^{2} u / d x^{2}\right)$ are the secondary variables, any four values of which are usually specified on the boundary, namely, $x=0$ or $x=L$.

The direct problem of the Euler-Bernoulli beam theory requires the determination of the deflection $u$ which satisfies eqn.(1) (or $M$ and $u$ satisfying eqns (2) and (3)), when $a>0$ and $f \geq 0$ are given and four boundary conditions (essential or natural) are prescribed. If at least one of these boundary conditions is on the deflection then this direct problem is well-posed. However, there are other problems (termed inverse) that may be associated with the Euler-Bernoulli eqns (1)-(3). For simply supported beams an inverse load identification problem, which requires the determination of the load $f(x)$ which satisfies eqn.(3) when $a$ and $M$ are given, has been investigated by Collins et al. (1994) in the practical context of recovering engineering loads from strain gauge data. This problem is ill-posed since it violates the stability of the solution, which relates to twice differentiating numerically $M$, which is a noisy function.

In this study, we investigate an inverse coefficient identification problem which requires the identification of the positive, heterogeneous flexural rigidity coefficient $a(x)$ of a beam which satisfies eqn.(1), (or eqns (2) and (3)), when $u$ and $f$ are given. This problem is an extension to higherorder differential equations of the inverse problem analysed by Marcellini (1982) for the one-dimensional Poisson equation. Prior to this study, a numerical algorithm has been recently proposed by Ismailov and Muravey (1996) for supported plates, when $f>0$ and $u \leq 0$. However, the theoretical investigation of the uniqueness of the solution is much more difficult since eqn.(1) does not, in general, have a unique solution, e.g. if $a_{0}$ is a solution of eqn.(1) then for any linear function $h$, the function $a_{1}=a_{0}+h\left(d^{2} u / d x^{2}\right)^{-1}$ may still be a solution of eqn.(1). Therefore, a necesary condition for the uniqueness of the solution of eqn.(1) is that the set

$$
\begin{equation*}
S=\left\{x \in[0, L] \left\lvert\, \frac{d^{2} u}{d x^{2}}(x)=0\right.\right\} \tag{4}
\end{equation*}
$$

is not empty. However, at the other extreme, if $S$ is a subset of $[0, L]$ of non-zero measure then $a(x)$ is not identifiable on $S$.

Further, based on the physical argument that the properties of the beam, namely $E$ and $I$, should be positive, and considering only applications in which there is always a positive finite load acting on the beam, we can assume that there are $0<f_{1} \leq f_{2}<\infty$ such that $f_{1} \leq f(x) \leq f_{2}$ and
that there are $0<\lambda_{1} \leq \lambda_{2}<\infty$ and $Q>0$ such that $a(x)$ belongs to the domain definition set

$$
\begin{equation*}
A=\left\{a \in L^{\infty}(0, L) \mid \lambda_{1} \leq a \leq \lambda_{2},\left\|a^{\prime}\right\| \leq Q\right\} \tag{5}
\end{equation*}
$$

where $\|$.$\| denotes the L^{2}(0, L)$-norm. For the present analysis we restrict ourselves to homogeneous boundary conditions to accompany the non-zero load $f$ and the linear eqns (1)-(3). Also, since $a>0$ then the set $S$ given by eqn.(4) is $S=\{x \in[0, L] \mid M(x)=0\}$. Thus boundary conditions which ensures that $S \neq \emptyset$ include $u^{\prime}(0)=u^{\prime}(L)$ or $M(0) M(L)=0$. If only Dirichlet and/or Neumann type boundary conditions for $u$ and $M$ are considered then there are in total $\sum_{k=0}^{4} C_{4}^{k} C_{4}^{4-k}=70$ of these possibilities. However, half of these possibilities are equivalent, based on the invariance of eqns (1)-(3) under the translation $x \mapsto(L-x)$. The aim of this study is not to investigate all these possibilities but rather select realistic physical boundary conditions associated with beams that naturally occur in elasticity, such as beams supported at both ends, namely,

$$
\begin{equation*}
u(0)=u(L)=M(0)=M(L)=0 \tag{6}
\end{equation*}
$$

Other types of boundary conditions have been investigated elsewhere, see Lesnic et al. (1999).

The plan of the paper is as follows. In section 2 we prove that the operator $u \mapsto a$ is Holder continuous and thus injective, i.e. the uniqueness and continuous dependence on the input data of the inverse problem. Further, when random noisy discrete input data is included, a regularization algorithm is developed in section 3 and the numerically obtained results are illustrated and discussed.

## MATHEMATICAL ANALYSIS

We formally define the operator $U: A \rightarrow L^{2}(0, L)$

$$
\begin{equation*}
U(a):=u_{a}=\iint \frac{M}{a}, \quad M=\iint f, \quad \forall a \in A \tag{7}
\end{equation*}
$$

as a formal general solution of eqns (1)-(3), where the integral sign $\int$ is understood in the sense that the constants of integration are to be determined by imposing the boundary conditions (6).

For convenience, and in order to simplify the algebraic manipulations, a uniform load $f \equiv 1$ was assumed which in turn gives

$$
\begin{gather*}
M(x)=\frac{x(x-L)}{2}  \tag{8}\\
u_{a}(x)=\int_{0}^{x} d t \int_{0}^{t} \frac{\tau(\tau-L)}{2 a(\tau)} d \tau-\frac{x}{L} \int_{0}^{L} d t \int_{0}^{t} \frac{\tau(\tau-L)}{2 a(\tau)} d \tau \tag{9}
\end{gather*}
$$

Lemma 1. The following inequality holds

$$
\begin{equation*}
\left\|u_{a}^{\prime \prime}-u_{b}^{\prime \prime}\right\|^{3 / 2} \leq\left\|u_{a}-u_{b}\right\|^{1 / 2}\left\|u_{a}^{\prime \prime \prime}-u_{b}^{\prime \prime \prime}\right\| \tag{10}
\end{equation*}
$$

Proof: To evaluate the $L^{2}$-norm $\left\|u_{a}^{\prime \prime}-u_{b}^{\prime \prime}\right\|$ we use integration by parts, impose the boundary conditions (6) and apply the Holder inequality to obtain

$$
\begin{gather*}
\left\|u_{a}^{\prime \prime}-u_{b}^{\prime \prime}\right\|^{2}=-\int_{0}^{L}\left(u_{a}^{\prime}-u_{b}^{\prime}\right)\left(u_{a}^{\prime \prime \prime}-u_{b}^{\prime \prime \prime}\right) d x \\
\leq\left\|u_{a}^{\prime}-u_{b}^{\prime}\right\|\| \| u_{a}^{\prime \prime \prime}-u_{b}^{\prime \prime \prime} \| \tag{11}
\end{gather*}
$$

Finally, combining the inequalities (11) and (12) yields the inequality (21).

Lemma 2. If $a \in A$ then there is a positive constant $C$ such that $\left\|u_{a}^{\prime \prime \prime}\right\| \leq C$.
Proof: If we denote $\alpha=a^{-1}, \alpha_{2}=\lambda_{1}^{-1}$ then we have $0<$ $\alpha \leq \alpha_{2}<\infty$ and $u_{a}^{\prime \prime}=\alpha M$. Thus $u_{a}^{\prime \prime \prime}=\alpha^{\prime} M+\alpha M^{\prime}$ and using eqn.(8) we obtain the following estimates for $\left\|u_{a}^{\prime \prime \prime}\right\|$, namely,

$$
\begin{equation*}
\left\|u_{a}^{\prime \prime \prime}\right\| \leq\left\|\alpha^{\prime}\right\| \frac{L^{2}}{8}+\alpha_{2}\left(\frac{L^{3}}{12}\right)^{1 / 2} \tag{13}
\end{equation*}
$$

Finally, since $\left|\alpha^{\prime}\right|=\frac{\left|a^{\prime}\right|}{a^{2}} \leq \alpha_{2}^{2}\left|a^{\prime}\right|$ and thus $\left\|\alpha^{\prime}\right\| \leq$ $\alpha_{2}^{2}\left\|a^{\prime}\right\| \leq \alpha_{2}^{2} Q$, from eqn.(13) it follows that there is a positive constant $C>0$ such that $\left\|u_{a}^{\prime \prime \prime}\right\| \leq C$, which concludes lemma 2.

Lemma 3. If $a, b \in A$ then there is a positive constant $C$ such that

$$
\begin{equation*}
\|a-b\|_{L^{2}} \leq C\left\|u_{a}^{\prime \prime}-u_{b}^{\prime \prime}\right\|_{L^{2}}^{2 / 3} \tag{14}
\end{equation*}
$$

Proof: If we denote $\alpha=a^{-1}, \beta=b^{-1}, \alpha_{1}=\lambda_{2}^{-1}$ and $\alpha_{2}=\lambda_{1}^{-1}$, then we have $0<\alpha_{1} \leq \alpha, \beta \leq \alpha_{2}<\infty$ and $u_{a}^{\prime \prime}=\alpha M, u_{b}^{\prime \prime}=\beta M$.

The key idea of the proof is based on removing (similar to a Cauchy finite part evaluation of singular integrals) the singularities of the function $\alpha=u_{a}^{\prime \prime} / M$ (and similarly $\beta=$ $\left.u_{b}^{\prime \prime} / M\right)$ which, based on eqn.(8), is of the type $x^{-1}(L-$ $x)^{-1}$. Therefore we extend the functions $u_{a}, u_{b}, \alpha$ and $\beta$ to the whole real axis by defining them to be zero outside of the interval $[0, L]$. Then, for any $\epsilon>0$, using the Holder inequality, we have the following identities and inequalities

$$
\begin{array}{r}
\|\alpha-\beta\|^{2}=\int_{(-\epsilon, \epsilon) \cup(L-\epsilon, L+\epsilon)}|\alpha-\beta|^{2} d x \\
+\int_{(-\infty,-\epsilon) \cup(\epsilon, L-\epsilon) \cup(L+\epsilon, \infty)}|\alpha-\beta| \frac{\left|u_{a}^{\prime \prime}-u_{b}^{\prime \prime}\right|}{|M|} d x \\
\leq 4 \epsilon \alpha_{2}^{2}+\alpha_{2}| | u_{a}^{\prime \prime}-u_{b}^{\prime \prime}| | \\
\left(\int_{(-\infty,-\epsilon) \cup(\epsilon, L-\epsilon) \cup(L+\epsilon, \infty)} \frac{4}{x^{2}(L-x)^{2}} d x\right)^{1 / 2} \\
=4 \alpha_{2}^{2} \epsilon+\frac{2^{3 / 2} \alpha_{2}}{L}\left\|u_{a}^{\prime \prime}-u_{b}^{\prime \prime}\right\| \\
\left\{\frac{2}{\epsilon}-\frac{2 \epsilon}{L^{2}-\epsilon^{2}}+\frac{2}{L} \ln \left(\frac{|L-\epsilon|}{|L+\epsilon|}\right)\right\}^{1 / 2} \\
\leq 4 \alpha_{2}\left(\alpha_{2} \epsilon+\| u_{a}^{\prime \prime}-u_{b}^{\prime \prime}| | \epsilon^{-1 / 2} / L\right) \tag{15}
\end{array}
$$

The right-hand side of expression (15), as a function of $\epsilon$, attains its minimum at

$$
\begin{equation*}
\epsilon_{\min }=\left(\frac{\left\|u_{a}^{\prime \prime}-u_{b}^{\prime \prime}\right\|}{2 \alpha_{2} L}\right)^{2 / 3} \tag{16}
\end{equation*}
$$

and introducing this value into expression (15) we obtain

$$
\begin{equation*}
\|\alpha-\beta\|^{2} \leq\left(2^{4 / 3}+2^{1 / 3}\right) L^{-2 / 3} \alpha_{2}^{4 / 3}\left\|u_{a}^{\prime \prime}-u_{b}^{\prime \prime}\right\|^{2 / 3} \tag{17}
\end{equation*}
$$

Finally, the estimate (14) is obtained using the inequality $|a-b| \leq \alpha_{1}^{-2}|\alpha-\beta|$ and eqn.(17).

At this stage, using lemmas $1-3$, we can conclude the following theorem:

Theorem 1. (The Holder continuity of the inverse operator $\left.U^{-1}(u)=a\right)$
If $a, b \in A$ then there is a positive constant $C$ such that

$$
\begin{equation*}
\|a-b\| \leq C\left\|u_{a}-u_{b}\right\|^{1 / 9} \tag{18}
\end{equation*}
$$

As an immediate consequence of this theorem we conclude the uniqueness and continuous dependence of the solution $a \in A$ on the input data $u$.

## REGULARIZATION ALGORITHM

In this section the solution of the inverse problem (1), which consists of finding $a \in A$ when the additional data

$$
\begin{equation*}
u_{a} \approx z \tag{19}
\end{equation*}
$$

is given, is based on the first-order regularization method of Tikhonov and Arsenin (1977), which requires the finding of the minimum with respect to $\alpha=a^{-1} \in A$ of the functional

$$
\begin{equation*}
\Phi_{\lambda}(\alpha)=\left\|u_{a}-z\right\|^{2}+\lambda\left\|\alpha^{\prime}\right\|^{2} \tag{20}
\end{equation*}
$$

where $\lambda>0$ is a regularization parameter to be prescribed.
For simply supported beams subjected to a unit load we can obtain the deflection $u(x)$ as the solution of the second-order differential equation

$$
\begin{equation*}
u^{\prime \prime}(x)=\frac{x(x-L)}{2} \alpha(x), \quad u(0)=u(L)=0 \tag{21}
\end{equation*}
$$

Integrating eqn.(21) twice we obtain

$$
\begin{equation*}
u(x)=\int_{0}^{L} G(x, s) \alpha(s) d s, \quad 0 \leq x \leq L \tag{22}
\end{equation*}
$$

where the Greens function $G(x, s)$ is given by

$$
G(x, s)= \begin{cases}\frac{s^{2}(s-L)(x-L)}{2 L}, & s \leq x  \tag{23}\\ \frac{s x(s-L)^{2}}{2 L}, & x \leq s\end{cases}
$$

However, the transformation of the differential eqn.(21) into the integral eqn.(22) does not eliminate the ill-posedness of the problem, as such an equation is a Fredholm integral equation of the first kind and constitutes a well-known example of an ill-posed problem due to its instability of the solution with respect to noise in the input data $z$ as given by eqn.(19), see e.g. Tikhonov and Arsenin (1977).

We now divide the interval $[0, L]$ into $N$ equal parts by putting $s_{j}=j L / N$ for $j=\overline{0, N}$, and on each subinterval $\left[s_{j-1}, s_{j}\right]$ for $j=\overline{1, N}$, we assume that $\alpha$ is constant and takes its value at the midpoint $\tilde{s_{j}}=\left(s_{j}+s_{j-1}\right) / 2$, i.e.

$$
\begin{equation*}
\left.\alpha\right|_{\left[s_{j-1}, s_{j}\right]}=\alpha\left(\tilde{s_{j}}\right)=\alpha_{j}, \quad j=\overline{1, N} \tag{24}
\end{equation*}
$$

Then eqn.(22) can be approximated by

$$
\begin{equation*}
u(x)=\sum_{j=1}^{N} \alpha_{j} \int_{s_{j-1}}^{s_{j}} G(x, s) d s, \quad 0 \leq x \leq L \tag{25}
\end{equation*}
$$

If the deflection $u(x)$ is recorded only at $K$ discrete locations $x_{i}=(2 i-1) L /(2 K)$ for $i=\overline{1, K}$, then eqn.(25) gives

$$
\begin{equation*}
u_{i}=u\left(x_{i}\right)=\sum_{j=1}^{N} A_{i j} \alpha_{j}, \quad i=\overline{1, K} \tag{26}
\end{equation*}
$$

where the integrals

$$
\begin{equation*}
A_{i j}=\int_{s_{j-1}}^{s_{j}} G\left(x_{i}, s\right) d s, \quad i=\overline{1, K}, \quad j=\overline{1, N} \tag{27}
\end{equation*}
$$

can be evaluated analytically and their expressions are given by

$$
\begin{gathered}
A_{i j}=(2 i-1)\left(12 j^{3}-18 j^{2}-24 N j^{2}+12 j+12 j N^{2}\right. \\
\left.\left.+24 N j-6 N^{2}-8 N-3\right) L^{4}\right) /\left(48 K N^{4}\right), \quad \text { if } i<j \\
\quad A_{i j}=(2 i-1-2 K)\left(12 j^{3}-18 j^{2}-12 N j^{2}+\right. \\
+12 j+12 N j-4 N-3) L^{4} /\left(48 K N^{4}\right), \quad \text { if } i>j
\end{gathered}
$$

$$
\begin{gathered}
A_{i j}=\left[(2 i-1) j^{2}\left(3 j^{2}-8 N j+6 N^{2}\right)\right. \\
\left.+(-2 i+1+2 K)(j-1)^{3}(3 j-3-4 N)\right] L^{4} /\left(48 K N^{4}\right) \\
+(2 i-1)^{3}(2 i-4 K-1) L^{4} /\left(384 K^{4}\right), \quad \text { if } i=j(28)
\end{gathered}
$$

In order to accommodate for the physical reality that in practice the measurements of $u$ will never be exact or smooth, the deflection $u$ is contaminated with some errors, say $\underline{\epsilon}=\epsilon\left(x_{i}\right)$ for $i=\overline{1, K}$, given by

$$
\begin{equation*}
\underline{z}=\underline{u}+\underline{\epsilon} \tag{29}
\end{equation*}
$$

Thus, we have transformed the infinite-dimensional illposed problem (22) into the finite-dimensional problem of finding the vector $\underline{\alpha}$ from the ill-conditioned system of linear equations

$$
\begin{equation*}
\underline{z} \approx \underline{u}=A \underline{\alpha} \tag{30}
\end{equation*}
$$

In order to deal with the ill-conditioned system of linear eqns (30), the minimization of the discretised version of eqn.(20) results in a stable numerical solution given by

$$
\begin{equation*}
\underline{\alpha}=\left(A^{t r} A+\lambda R\right)^{-1} A^{t r} \underline{z} \tag{31}
\end{equation*}
$$

where $R$ is the first-order regularization matrix which has the components $R_{i i}=1$ for $i=1$ and $i=K, R_{i i}=2$ for $i=\overline{2,(K-1)}, R_{i(i+1)}=-1$ for $i=\overline{1,(K-1)}, R_{i(i-1)}=$ -1 for $i=\overline{2, K}$ and $R_{i j}=0$ otherwise.

For a simple typical benchmark example, namely that of a simply supported beam of length $L=1$ having a deflection $u(x)=\frac{x^{5}}{20}-\frac{x^{3}}{6}+\frac{7 x}{60}$, with the flexural rigidity coefficient to be retrieved given by $a(x)=\frac{1}{2(x+1)}$, the solution given by eqn.(31) has been employed. Also, in order to investigate the stability of the numerical solution, $p \%=1 \%$ noise has been included in the measured deflection $\underline{z}$, as given by eqn.(29), where the components of the noise vector $\underline{\epsilon}$ were generated using the NAG routine G05DDF, as Gaussian random variables with mean zero and standard deviation

$$
\begin{equation*}
\sigma=\frac{p}{100} \times \max |u(x)|=\frac{p}{100} \times 4 \times 10^{-2} \tag{32}
\end{equation*}
$$

The numerically obtained results for $a(x)=\alpha^{-1}(x)$ given by eqn.(31), when $K=N=20$ and $p=0$ and $\lambda=0$, and when $p=1$ and $\lambda=5 \times 10^{-5}, 10^{-5}, 5 \times 10^{-6}$ and $10^{-6}$, are shown in Fig. 1 and also for comparison the exact solution is presented. From this figure it can be seen that when no noise is included in the data, i.e. $p=0$, then a simple inversion $\underline{\alpha}=A^{-1} \underline{z}$ retrieves very accurately the exact solution.


Figure 1. THE NUMERICAL RESULTS (-) FOR VARIOUS VALUES OF $\lambda$ WHEN $p \%=1 \%$, IN COMPARISON WITH THE EXACT SOLUTION $a(x)=\frac{1}{2(1+x)}(---)$. THE CIRCLES (0) REPRESENT THE NUMERICAL RESULTS OBTAINED WHEN $p=0$ AND $\lambda=0$.

However, when noise is included in the data then as $\lambda$ becomes very small, say $\leq 10^{-6}$, oscillations start to develop in the numerical solution which becomes unstable. Also as $\lambda$ becomes large, say $>10^{-4}$, then the numerical solution tends to a constant value. However, there is a wide range of values of $\lambda$, say $5 \times 10^{-5}<\lambda<5 \times 10^{-6}$ for which the estimate of the solution is stable and reasonably accurate. Nevertheless there is quite a significant disagreement between the exact and the numerical solutions near the ends of the beam, and in fact the numerical solution may be considered a good estimate of the exact solution only in the range $0.2<x<0.8$.

Finally, it is reported that alternative numerical methods employed by the author, which are based on the singular value decomposition or the mollification method, also failed to produce accurate estimates of the solution near the ends of the fixed beam. In a future study it is proposed that more powerful numerical methods which are based on parameterisation and/or constrained minimization procedures should be employed. Also, future work will be concerned with the inverse coefficient identification problem associated with the Euler-Bernoulli unsteady-state beam theory.

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