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EXP05

OPTIMAL EXPERIMENTAL DESIGN WITH INVERSE PROBLEMS

Mikhail Romanovski
CAD/CAE Department, POINT Ltd
22/30, ave. Andropova, Moscow, Russia
E-mail: pointltd@glasnet.ru

ABSTRACT

The main goal is a theory development of optimal experimental design for the heat properties identification of a solid. Sequential and methodical study of identification of a specific heat and thermal conductivity are carried out. The wide range of mathematical models are considered. The model choice is determined by possibilities of analytical analysis of design conditions.

The investigation answers to a number of principal questions of thermophysical experimental realization and design. Their practical significance is expressed in discovering the existence of the experimental informatibility upswing even of very small sample volume.

NOMENCLATURE

x	Spatial coordinate
t	Time
$u(x,t)$	Temperature field
f	Volume heat sources
u_0	Initial temperature
$v_{0,1}$	Boundary temperatures
a_1	Specific heat coefficient
a_2	Thermal conductivity coefficient
ε	Noise of observations
δ	Absolute error of observations
u^δ	Sample of observations
$\bar{u}$	Prototype state
ν	Absolute error of identification
μ	Relative error of identification
Ξ	Optimal design
Ω	Stabilizing functional
$\theta_{1,2}$	Form-factors of identification errors mode
$\mathcal{R}$	Indeterminacy power of model identification

INTRODUCTION

At the moment, despite a significant history of experimental heat exchange researches, the theory of its optimal design has not answers to the whole series of basic in essence problems. The known solutions [1-3] were considered for cases, which don't allow to receive a full picture of optimal design conditions.

We shall deliver and study the following questions.

What is the essence of optimal observations design: in searching of locally-optimal points, individual for each specimen and conditions of its thermal loading or in existence of the common circuit of observations, identical for any experiment, but updated on a number of conditions?

How to find identification errors dependence vs observations allocation for any kind of an experiment?

Which measurements guarantee a minimum level of identification errors of heat transfer properties? Which factors and conditions of experimental realization ensure a decrease of an identification errors level?

Are significant functional features like symmetry of observations, heterogeneity of temperature distribution, ratios between object properties, ratios between boundary conditions and others for an optimal identification?

Whether statistical indeterminacy of observations has an influence to an identifiability of an experiment?

Is it possible to find as heat properties of specimen and its boundary conditions using only one point of observation?

These questions on the whole bring out the principal character of design peculiarities. Their solutions are carried out below. The ones demonstrate the tools for the informatibility analysis of a complex experiment.

EXPERIMENTAL DESIGN METHOD

The determination of optimal observations must be carried out in general with allowance for the methodological characteristics of inverse problem. This can be accomplished to the fullest extent by means of the Tikhonov regularization principle [4]. We use such regularization below, invoking a special procedure [5]. The one allows to achieve the best fit to the observations and doesn't require a regularization parameter.

Essential feature of the design method is a type and character of obtained estimations. This investigation is based on idea of *a guaranteed error* [6]. In this case a minimization of identification error

$$\nu_{rms} = \sqrt{\sum_{i=1}^{n} \nu_i^2 / n}\ .$$

is carried out. In addition to the *rms* norm, which provides a means for analyzing observations from the standpoint of total error, other forms of estimation error can be considered. The most practical form here could be the absolute-error estimator. The one requires a minimization of the error

$$\nu_{abs} = \max_i |\nu_i|\ .$$

Let us define the factors of experimental realization, which influences to the accuracy solution of inverse problem.

Definition 1. *An indeterminacy power of mathematical model identification is a factor determining a level of identification error in accordance to the measurements errors and the type and strength of a driving force.*

The one is expressed a main conditions due to identification errors is decreased to zero. To separate the conditions of experimental realization which has an influence to the sensors allocations it is necessary to introduce

Definition 2. *A form-factor of identification error mode is a factor determining the character of identification errors distribution vs sensors allocations.*

A significance of these two type of factors and their comparison will be shown below.

ONE UNKNOWN COEFFICIENT

We specify the mathematical model

$$\frac{du}{dt} = a(u_{am} - u), \quad t > 0;$$
$$u\big|_{t=0} = u_0 \qquad (1)$$

with an unknown coefficient $a = const>0$.

The observation fitting equation [6] for the model (1) is expressed in the final form

$$\max_t |(u_{am} - u_0)\exp(-\bar{a}t)[\exp(\mu\bar{a}t) - 1] + \varepsilon| = \delta$$

where $\mu = (a - \bar{a})/\bar{a}$ is the relative identification error, $\bar{a}$ is the true value of the required coefficient. The identification error is determined by the expression

$$\mu = \frac{1}{\bar{a}t}\ln[1 + 2\mathcal{R}\exp(\bar{a}t)]\ ,$$

The corresponding graphs are show in Fig.1. Here

$$\mathcal{R} = \frac{\delta}{u_{am} - u_0}$$

is the indeterminacy power of the model (1) identification. The one determines main factors and conditions due to an identification error level is established and reduced.

For any noise level $||\varepsilon|| < \infty$ the coefficient a will be determined with optimal error $\mu^{(C)} = \min_t \mu(t)$ at the time

$$t^{opt} = \frac{1}{\bar{a}\mu^{(C)}}\ln\frac{1}{1 - \mu^{(C)}}$$

The *R*-optimal and *C*-optimal designs [6] for the model (1) identification are equivalent. The final expression for the minimum guaranteed identification error is represented in implicit form

$$\mu^{(C)}(1 - \mu^{(C)})^{\frac{1-\mu^{(C)}}{\mu^{(C)}}} = 2\mathcal{R}$$

The behavior of the guaranteed error and the character of the optimal observation time have the next peculiarities. First, the identification error of the model (1) depends on the noise level and the difference between initial and ambient temperatures. Consequently, the condition for offsetting growth of the observation errors and diminishing their influence is to increase the temperature difference. Second, the upper bound of the indeterminacy power of the model (1) identification is $\mathcal{R}_{\max} = 1/2$. Experiments, which have $\mathcal{R} \geq \mathcal{R}_{\max}$, cannot to guarantee an identification accuracy. Third, the lower bound of the optimal measurement time is the value $T_{min} = 1/\bar{a}$. This value establishes a certain barrier, below which observations are not recommended, i.e., $t^{opt} > T_{min}$.

Let the state of an object be described by following model

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \ell, \ t > 0;$$
$$u\big|_{t=0} = u_0, \qquad 0 < x < \ell;$$
$$u\big|_{x=0} = v_0,\ u\big|_{x=\ell} = v_1, \qquad t > 0 \qquad (2)$$

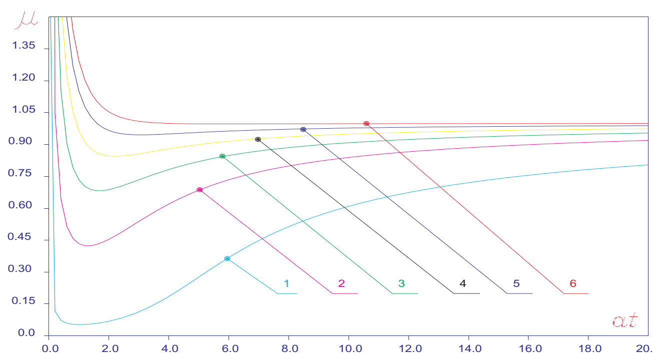

Figure 1. IDENTIFICATION ERROR OF MODEL (6)

1 - $\mathcal{R}$=0.01, 2 - $\mathcal{R}$= 0.1, 3 - $\mathcal{R}$=0.2, 4 - $\mathcal{R}$= 0.3, 5 - $\mathcal{R}$= 0.4, 6 - $\mathcal{R}$=0.49

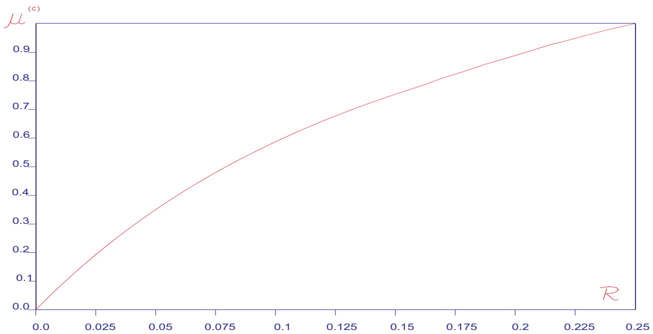

Figure 2. *C*-OPTIMAL IDENTIFICATION OF MODEL (11)

In case $v_{0,1} = const$ the observation fitting equation is expressed as

$$\max_{x,t}\left|\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k(v_1-u_0)-(v_0-u_0)}{k}\sin\frac{k\pi}{\ell}x\times\right.$$
$$\left.\times\exp\left[-\bar{a}\left(\frac{k\pi}{\ell}\right)^2 t\right]\left\{1-\exp\left[\mu\bar{a}\left(\frac{k\pi}{\ell}\right)^2 t\right]\right\}+\varepsilon\right|=\delta \qquad (3)$$

where $\mu=(\bar{a}-a)/\bar{a}$. From Eq.(3) follows that for any $\bar{a}$ and $\|\varepsilon\|<\infty$ the minimum identification error is attained for the optimal observation allocation $x^{opt} = \ell/2$. The time dependence of the identification error μ is expressed by the equation

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{2k-1}\exp\left[-\bar{a}\left(\frac{(2k-1)\pi}{\ell}\right)^2 t\right]\times$$
$$\left\{1-\exp\left[\mu\bar{a}\left(\frac{(2k-1)\pi}{\ell}\right)^2 t\right]\right\}=\pi\mathcal{R},$$

where

$$\mathcal{R}=\frac{\delta}{v_1+v_2-2u_0}$$

is the indeterminacy power of the model (2) identification. The appropriate optimal time is expressed as local-optimal

$$t^{opt}=Arg\min_{t}\mu(t)$$

The dependence of *C*-optimal identification error from the conditions of experimental realization is represented in Fig.2.

The model (2) has following identification peculiarities. The lower bound of the optimal measurement time is $T_{\min}=\ell^2/\pi^2\bar{a}$. The upper bound of the indeterminacy power of the model (11) identification is $\mathcal{R}_{\max} = 1/4$.

TWO UNKNOWN COEFFICIENTS

Let us specify the next mathematical model

$$a_1\frac{\partial u}{\partial t}=a_2\frac{\partial^2 u}{\partial x^2}+f,\quad 0<x<\ell, t>0;$$
$$u\big|_{t=0}=u_0,\quad 0<x<\ell;$$
$$u\big|_{x=0}=v_0,\quad u\big|_{x=\ell}=v_1,\quad t>0 \qquad (4)$$

It is required to find such pair of points $\Xi=\{x_i^{opt},t_i^{opt}\}_{i=1,2}$ in which known sample of observations

$$u_i^{\delta}=\bar{u}(x_i,t_i)+\varepsilon_i,\quad i=1,2$$

can to define a specific heat a_1 and a thermal conductivity a_2 with a minimum guaranteed error on *rms*

$$\mu^{(R)}=\min_{x,t}\sqrt{\frac{\mu_1^2+\mu_2^2}{2}}$$

or minimax

$$\mu^{(C)}=\min_{x,t}\max_{i}|\mu_i|,$$

criteria. Here $\mu_{1,2}=(\bar{a}_{1,2}-a_{1,2})/\bar{a}_{1,2}$ are relative errors of identification, $\bar{a}_{1,2}$ are unknowns defined the true state $\bar{u}$, ε is measurements noise.

We shall accept a number of the suppositions. At first, we shall consider $a_{1,2}=const$. Besides a specimen density is included as known constant in coefficient a_1. Secondly, we shall limit an aspect of thermal loading conditions to values f, $u_0, v_{0,1} = const$. These restrictions will allow to simplify the analysis of optimal designs dependence from conditions of experimental realization. At the same time these restrictions do not reduce number of ratios between boundary conditions. They envelop a broad band of practical cases and all characteristic ratios between thermal loading are reflected.

Note, that the volume of observations $\{u_i^{\delta}\}_{i=1,2}$ is selected from a condition of supporting of its admissible minimum. This restriction of sample volume allows to analyze the achievement of maximum experimental informatibiality, rested minimum initial information [8].

The condition of the model state fitting with observations is given by system

$$\max_{x_i,t_i}\left|\sum_{s=1}^{3}\bar{F}_s(x_i,t_i)-\sum_{s=1}^{3}\hat{F}_s(x_i,t_i)+\varepsilon_i\right|=\delta,\quad i=1,2, \qquad (5)$$

where

$$F_1=\frac{2}{\pi^3}\frac{f\ell^2}{a_2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3}\times$$
$$\times\left\{1-\exp\left[-\frac{a_2}{a_1}\left(\frac{(2k-1)\pi}{\ell}\right)^2 t\right]\right\}\sin\frac{(2k-1)\pi}{\ell}x$$

$$F_2=\frac{2}{\pi}[2u_0-(v_0+v_1)]\sum_{k=1}^{\infty}\frac{1}{2k-1}\times$$
$$\times\exp\left[-\frac{a_2}{a_1}\left(\frac{(2k-1)\pi}{\ell}\right)^2 t\right]\sin\frac{(2k-1)\pi}{\ell}x$$

$$F_3=\frac{(v_1-v_0)}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\exp\left[-\frac{a_2}{a_1}\left(\frac{2k\pi}{\ell}\right)^2 t\right]\sin\frac{2k\pi}{\ell}x$$

The functions $\bar{F}_s$ are represented by true values $\bar{a}$ of unknowns, and the functions $\hat{F}_s$ are expressed by values $\hat{a}$, obtained in an outcome of identification.

Optimal designs

Let us study a behavior of functions $\mu_{1,2}$ at variations of their variables x,t. A term by term comparison of expression, obtained by a difference for any $t' < t''$ one of the equations (5) allows to find the following conditions

$$E' - E'' > 0,$$

$$\frac{1}{1-\mu_2'} E' - \frac{1}{1-\mu_2''} E'' > 0,$$

where

$$E = \exp\left[-\frac{\bar{a}_2(1-\mu_2)}{\bar{a}_1(1-\mu_1)} \left(\frac{k\pi}{\ell} \right)^2 t \right]$$

From them it follows that

$$\frac{1-\mu_2'}{1-\mu_2''} < 1.$$

By virtue of it for any $t' < t''$ and $\mu_2 < 1$ a condition $\mu_2' > \mu_2''$ takes place. It means, that the function μ_2 is monotonically decreasing on time. Then the least identification error of a thermal conductivity can be reached only at the stationary state observation, i.e., $t_1^{opt} = T_\infty$.

Accordance to this fact for the optimal time of measurements the spatial dependence of the function μ_2 from a sensor coordinate is expressed as

$$\mu_2\big|_{T_\infty} \equiv \min_t \mu_2 = \frac{1}{1 \pm \frac{|f|(\ell x - x^2)}{4\delta\bar{a}_2}} \qquad (6)$$

The one is obtained in case of absolute norm using for the model state fitting with observations.

From expression (6) we define the least guaranteed error $\mu_2^- = \min_x \mu_2\big|_{T_\infty}$. It is reached in the point $x_1^{opt} = \ell/2$ and matters

$$\mu_2^- = \frac{16\mathcal{R}}{16\mathcal{R}-1}, \qquad (7)$$

where

$$\mathcal{R} = \frac{\delta\bar{a}_2}{|f|\ell^2}$$

is the indeterminacy power of the model (4) identification. Any experiments cannot to guarantee an identification accuracy in case $\mathcal{R} \geq \mathcal{R}_{\max} = 1/32$, because $\mu_2^- > 1$.

We shall remark that the function $\mu_2\big|_{T_\infty}$ has the second minimum

$$\mu_2^+ = \frac{16\mathcal{R}}{16\mathcal{R}+1}.$$

The one for $\mathcal{R}>0$ does not express guaranteed level of thermal conductivity identification, but this error is necessary for taking into account hereinafter at error minimization.

The character of the error μ_1 is determined by a stage of the thermal process and ratios between a source function and boundary conditions $\{f, u_0, v_{0,1}\}$.

On an initial stage of the process, when $t\sim 0$, sharp growth of values μ_1 is supposed, since the exponential terms in functions $F_{1,2,3}$ linearly depend from t. Then the restriction of growth of exponential terms of the series (15) is reached by a choice such x, for which as the function $\sin(2k-1)\pi\, x/\ell$, and $\sin(2k\pi\, x/\ell)$ approach to zero. It means, that on an initial stage of the thermal process the error μ_1 is descreased when observations asymptotically approach to the boundaries of a specimen, $x \to 0, x \to \ell$. In themselves boundary $x = 0, x = \ell$, the function μ_1, as follows from (15), suffers a discontinuity of the second kind. According to this fact the designs

$$\Xi\big|_{u_0 = v_1} = \begin{Bmatrix} x_1^{opt} \to 0, & t_1^{opt} \to 0 \\ x_2^{opt} = \frac{\ell}{2}, & t_2^{opt} = T_\infty \end{Bmatrix},$$

$$\Xi\big|_{u_0 = v_0} = \begin{Bmatrix} x_1^{opt} \to \ell, & t_1^{opt} \to 0 \\ x_2^{opt} = \frac{\ell}{2}, & t_2^{opt} = T_\infty \end{Bmatrix}$$

are as *R*-, and *C*-optimal if the condition for example is fulfilled

$$|2u_0 - (v_0 + v_1)| = |v_1 - v_0|$$

and the function f does not exceed a value, which installs a significance of terms of function F_1 in (5).

In case $t>>0$ the restriction of μ_1 growth at the expense of parameters variations in exponential terms (5) is unsufficient. Then an identification error decreasing can be reached, if the observation is fulfilled in a point, where there is the maximum of harmonics of the series (5).

In view of the series (5) convergence only first terms as significant can to ensure the restriction of μ_1 growth. Therefore the extreme of μ_1 is localized in areas, which centers should place in one of points $\{0.25\ell; 0.5\ell; 0.75\ell\}$. A deviation of optimal coordinate from indicated points will be less significant with thermal diffusivity $a = a_2/a_1$ growth because a convergence velocity of the series (5) depends from a value of a thermal diffusivity.

The time for the second optimal observation in these cases is determined as locally-optimal

$$t^{opt} = Arg \min_{x,t} \max(\mu_1\big|_{\mu_2^-}, \mu_1\big|_{\mu_2^+})$$

Characteristic cases of *rms* error dependence vs sensor allocation minimized on time are shown in Fig.3 and 4.

It should be mentioned, that all optimal designs are obtained despite essential nonlinearity of a model state dependence from unknowns, and in the supposition of any nature of measurements noise. This result shows that even essentially nonlinear *design problem supposes its aggregate and has strictly defined circuit of measurements.*

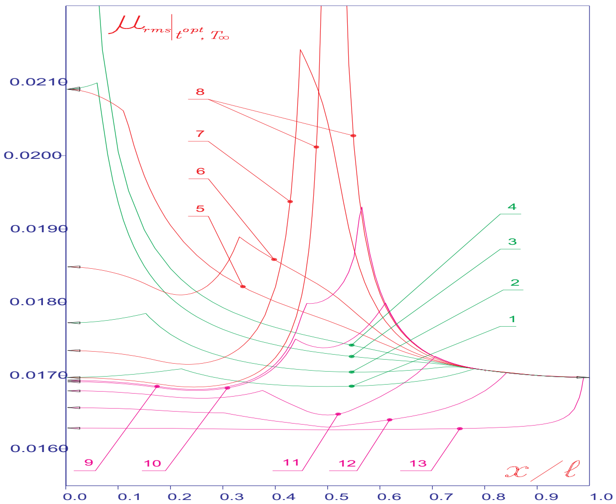

Figure 3. *RMS* ERROR DEPENDENCE: v_0 - *var*, $v_1<u_0$, $v_0^{(1)}<v_0^{(2)}<v_0^{(3)}<...$, 4-$u_0=v_0$, 8-$u_0=(v_0+v_1)/2$

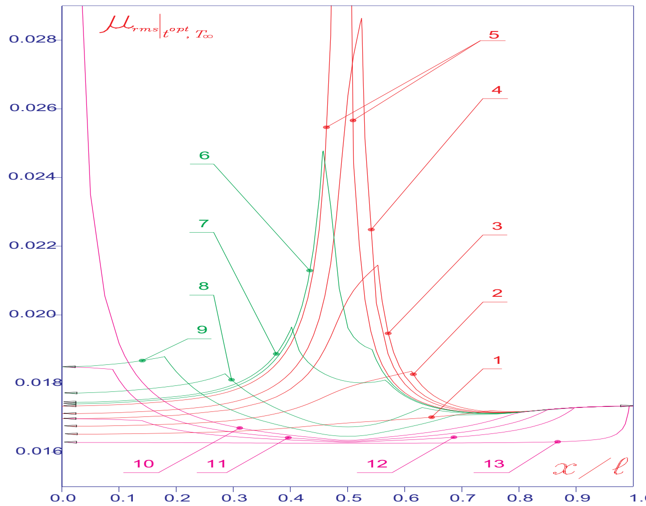

Figure 4. *RMS* ERROR DEPENDENCE: v_0 - *var*, $v_1>u_0$, $v_0^{(1)}<v_0^{(2)}<v_0^{(3)}<...$, 5-$u_0=(v_0+v_1)/2$, 10-$u_0=v_0$

Let us consider optimal design of the controlled experiments. It should be assumed, that the thermal loading is not fixed, and one is modified in desirable direction. There is the thermal loading condition

$$v_0 = v_1 \tag{8}$$

which may be to define as *the best guaranteed condition*. The one means independence of a position of the optimal point x_2^{opt} from ratio between factors $\{f,\ u_0,\ v_{0,1}\}$.

For a determination of an optimal observation at realization of the condition (8) we shall subtract one from other the equation (5) for different values x' and x''. We obtain

$$\exp\left\{-\frac{\bar{a}_2}{\bar{a}_1}\left[\frac{(2k-1)\pi}{\ell}\right]^2 t\right\}\left[\sin\frac{(2k-1)\pi}{\ell}x' - \sin\frac{(2k-1)\pi}{\ell}x''\right] =$$

$$= \exp\left\{-\frac{\bar{a}_2(1-\mu_2)}{\bar{a}_1(1-\mu_1')}\left[\frac{(2k-1)\pi}{\ell}\right]^2 t\right\}\sin\frac{(2k-1)\pi}{\ell}x' -$$

$$\exp\left\{-\frac{\bar{a}_2(1-\mu_2)}{\bar{a}_1(1-\mu_1'')}\left[\frac{(2k-1)\pi}{\ell}\right]^2 t\right\}\sin\frac{(2k-1)\pi}{\ell}x''$$

From it follows

$$\left|\sin\frac{(2k-1)\pi}{\ell}x'\right| > \left|\sin\frac{(2k-1)\pi}{\ell}x''\right|$$

Then at any $\{f,\ u_0,\ v_{0,1}\}$ for the error μ_1 minimization the observations would be selected when $|\sin(2k-1)\pi x/\ell|$ have a maximum. From all such points only one $x=\ell/2$ ensures a required maximum in each term of the series (5).

Therefore, if the condition (8) take place then the second optimal observation coincides with first point. The middle of a specimen appears the unique point, measurements in which guarantee the identification of heat properties with a minimum error. The appropriate optimal design is represented as

$$\Xi|_{v_0=v_1} = \{\ell/2, t^{opt}, T_\infty\}$$

Thus the analysis of extreme properties of the observation fitting equation (5) allows to specify in analytical form the wide range of design characters.

Basic properties of optimal design

The model (4) investigation has allowed to place the following features of heat properties identification.

1. The optimal design is expressed by the strictly defined circuit of measurements of a specimen temperature field.

2. In general the representation of a volume heat source, boundary temperatures and two interior observations are enough for identification as constant specific heat and thermal conductivity.

3. The consideration of individual properties of a specimen and its thermal loading does not change this circuit, but requires to make more accurate the position of one of the sensor and time of a measurement. A noise level does not influence to an aspect of the optimal design, but defines a magnitude of the identification errors.

4. The optimal position of the first interior observation does not depend on a character of thermal loading of a specimen and its heat properties. A sensor should be installed in the middle of a specimen.

5. The optimal observation time into the middle of a specimen for any experiments is a measurement of a stationary temperature of a specimen.

6. The optimal allocation of the second interior observation is localized in five limited areas. The significance any of them depends on conditions of experimental realization.

7. There is the broad band of boundary conditions variations, at which the position of the second interior observation is optimum only near to one of specimen boundaries, and the best time of measurements is determined

by the beginning of the heat process. The similar experiments are characterized by low difference between initial and one of the boundary temperatures.

8. Optimal observations may be carried out in one of the points $\{0.25\ell; 0.5\ell; 0.75\ell\}$ or their neighborhoods. The range of a deviation from these points is determined by magnitude of a specimen thermal diffusivity and ratios between boundary conditions. The optimal time of second measurement can be found as locally-optimal.

9. A design of the conditions of experimental realization allows to reduce volume of the observations to its minimum. The requirement of the boundary temperatures equality among themselve is the important condition for the measurements decreasing. In this case the number of observations for the model (4) identification is defined only as one interior point (not counting observations for deriving boundary conditions).

EXPERIMENTAL PECULIARITIES

Now we shall give attention to the influence of the experimental conditions on an identification accuracy.

Identifiability violation

The uniqueness solution of the system (5) requires a samples getting, which does not reduce equations to the linear dependence in points $\{x_i, t_i\}_{i=1,2}$. The one leads to the following condition

$$\exp\left[-\frac{\bar{a}_2}{\bar{a}_1}\left(\frac{k\pi}{\ell}\right)^2(t_1-t_2)\right]=\lambda\frac{\sin x_2\frac{k\pi}{\ell}}{\sin x_1\frac{k\pi}{\ell}},\ k=1,2,\ldots,\lambda\neq 0. \qquad (9)$$

Then the model (4) state is an unidentifiable on discrete sample u_*^δ only when the observations will be carried out in any two points $x^*_{1,2}$ such, that $x^*_1 + x^*_2 = \ell$, and the measurements will be executed at the same moment, $t_1 = t_2$.

The existence of unidentifiable state of the model (4) was shown earlier in [11]. The condition (9) expresses the additional unidentifiability, when the one-to-one correspondence is away only in separate points.

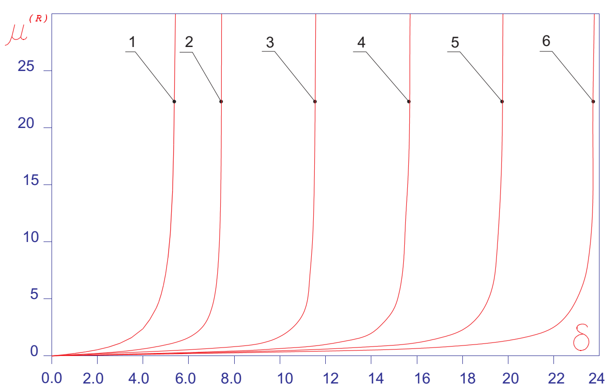

Figure 5. THRESHOLD LEVEL OF NOISE
1-f=50, 2-f=100,3-f=200,4-f=300,5-f=400,6-f=500

Threshold level of noise

As it is possible to see if $\mu_1 \to \infty$ then the system (5) is degenerated. Then the usage of observations, for which $\delta>\delta^*$, is meant a violation of domain of admissible values of system (5). By virtue of it the *threshold level of noise* δ^* is existed. Exceeding of this level will not allow to solve inverse problem.

The reason of the identifiability violation in this case is a degeneration of sample approximation by the model state. The one appears independent from coefficient a_1. A behavior of R-optimal error at variations of a noise level is represented in Fig.5.

The existence of a threshold δ^* testifies, that there is a value $f_1{}^*$, lower from which $f<f_1{}^*$ it is impossible to guarantee a satisfactory identification. The conditions of experimental realization, including function $f_1{}^*$, which caused a loss of an identifiability due to exceeding of some level of measurements noise, we shall define as *threshold conditions*.

Singular observations

From (6) follows, that for any f it is possible to specify such points $0<x^*<\ell$ in which the minimization of an identification error on time does not allow to get restriction of its values, i.e., $\mu_2|_{x^*,T_\infty}\to\infty$. The determination of heat properties on such observations is impossible, and temperature measurements in points

$$x^*=\frac{\ell}{2}\pm\frac{\ell}{2}\sqrt{1-16\mathcal{R}}$$

cannot to identify unknowns $a_{1,2}$. Measurements in the optimal point $x=\ell/2$, fulfilled in experiment which satisfies to the condition $\mathcal{R}=1/16$, not allow to find heat properties.

Such violation of identification difference from as an unidentifiability in a whole and in a small [7,8]. As it appears the existence of some errors corridor reduces a choice of thermal conductivity to arbitrary value. It is possible to specify such $f_2^*=16\delta a_2/\ell^2$, for which the required fitting between observations and model states in limits of a specific errors corridor is reached, if a line, if the boundary temperatures is connected by line. In an outcome the functional representation of a model state is degenerated and doesn't depend from a coefficient a_2. If $f\neq f_2^*$, then the approximation of observations depends on a_2.

We shall mark also existence of the low bound of a heat source strength. As it is follows from (7) the use $|f|<f_2^*$ reduces the identification error to values $|\mu_2^-|>>1$. Consequently it is impossible to guarantee satisfactory identification.

For the determinancy, and also underlining the special influence of measurements error to full loss of an identifiability, it is offered to distinguish observations as *an δ-unidentifiable*. Conditions of experiment, and in particular function $f_2{}^*$, generating unlimited growth of identification error under degeneration of a sample approximation, we shall define as *a singular*.

Self-compensating conditions of thermal loading

It is always can to specify the value $f = F^*$, for which the sum of negative terms of F_1 will be of one order of the sum of positive terms F_2 ($F_3 = 0$). Then at values f variations the considerable growth of exponential terms of the system (5) is possible in consequence of self-compensating of factors $\{f, u_0, v_{0,1}\}$. In this connection it is necessary to mean, that in general there is not monotone decrease of a minimum of the error μ_1^- at magnification of a value f.

In case $\{v_0+v_1 < u_0,\ v_0=v_1,\ f>0\}$ the evaluation is

$$F^* \sim \frac{\pi^2}{2}\frac{\bar{a}_2}{\ell^2}(u_0 - v_{0,1})$$

The one expresses conditions of emerging of the function μ_1^- maximum at variations $f > f_{1,2}^*$. The unique factor, permitting to limit a growth of the error μ_1^- is variations of observation time near the initial state.

The thermal loading detection, reducing to the identification accuracy loss in consequence of self-compensating of factors $\{f, u_0, v_{0,1}\}$, is necessary to consider as the important design feature. We shall define similar conditions as *self-compensating loadings*.

Factors of identification errors reduction

There are the following factors, which allow to decrease the identification errors.

At first, the decrease is reached at the expense of a diminution of noise level. From (5) implies, that if $\delta \to 0$, then $\mu_{1,2} \to 0$.

Secondly, from (7) it is followed, that for a diminution of magnitude μ_2 it is necessary to increase a strength of heat source. In according to the existence of δ-unidentifiable and the self-compensatings conditions it is necessary to require, that $f > \max(f_1^*, f_2^*)$ and $f\big|_{v_0=v_1} \neq F^*$. There is the value $f_g = 16(\delta + u_{\max})a_2/\ell^2$ since which C-optimal identification error of thermal conductivity will not exceed of a relative level of measurements noise

$$\left|\mu_2^-\right\|_{f\ge f_g} \le \delta / u_{\max},\ u_{\max} = \max(u_0, v_{0,1}).$$

Thirdly, a ratio between initial and boundary temperatures $\{u_0, v_{0,1}\}$ influences to the mode of identification errors. Magnification of a difference between u_0 and v_0+v_1 diminishes function μ_1. If $v_{0,1} \to \infty$, then we obtain

$$\exp\left\{-\frac{\bar{a}_2(1-\mu_2)}{\bar{a}_1(1-\mu_1)}\left[\frac{(2k-1)\pi}{\ell}\right]^2 t\right\} \to \exp\left\{-\frac{\bar{a}_2}{\bar{a}_1}\left[\frac{(2k-1)\pi}{\ell}\right]^2 t\right\}.$$

The one shows, that at magnification of temperature shock between the initial and boundary temperatures the function μ_1 tends to the value μ_2^-. From it follows that a temperature shock doesn't allow to achieve absolute decreasing of identification errors to zero. This result establishes, that for the model (4) the temperature shock as a factor of an identification errors decreasing has only local significance.

Whereat such functional features of temperature fields as the points of their maximum, inflection and others don't explicitly determine an optimal allocation of measurements.

Let us express an operation of the indicated factors as some generalized complexes. The system (5) has the dimensionless variables $\tau = a_2 t/(a_1\ell^2)$ and $\xi = x/\ell$. Therewith dimensionless groups

$$\theta_1 = \frac{2u_0 - (v_0+v_1)}{f\ell^2}a_2,\ \theta_2 = \frac{v_0 - v_1}{f\ell^2}a_2$$

can be introduced. The ones determine a mode of identification errors (Fig.3,4). By virtue of it the factors $\theta_{1,2}$ may be defined as *form-factors of identification error mode.*

A variation of the mathematical model parameters according to the conditions

$$\mathcal{R} = item,\ \theta_{1,2} = item \qquad (10)$$

and a choice of appropriate τ and ξ don't change a solution of the system (5). It means, that the errors $\mu_{1,2}$ are invariant relative to the conditions (10) of experimental realization.

Singularities of experimental realization

1. Major factors of the identification error decreasing are the raise of the heat source strength and the magnification of the difference between initial temperature on the one hand and boundary temperatures with other.

2. From two called factors the most significant is the magnification of a heat source strength. For any specific noise level the heat source can be indicated, since which it is guaranteed that an identification error always will be less then a noise level.

3. At the same time boundless magnification of a difference between initial and boundary temperatures doesn't allow to reduct the identification error to zero. In this case an asymptotic approach to the guaranteed identification error of thermal conductivity is reached only.

4. At variations of conditions of experimental realization and in particular magnification of a heat source strength it is necessary to take into account a presence of some singularities cases.

A threshold level expresses a limiting measurements noise. A mathematical model is unidentifiable when a sample has measurements error higher then threshold level.

A singular observations reduce to an identifiability loss thereof an arbitrary choice of unknowns at a certain breadth of a measurements error corridor.

A self-compensating loadings represents a significant worsening of the identification accuracy because of a counteraction each other of a specimen heat loading factors.

5. Alongside with uniqueness of a simultaneous determination of all heat properties an unidentifiability temperature fields as a whole, and in a small are existed.

SIMULTANEOUS IDENTIFICATION OF HEAT PROPERTIES AND BOUNDARY CONDITIONS

As it is proved above, the design of conditions of experimental realization allows to reduce a volume of sample to the minimum number of observations. The similar finding are characteristic for the approach of observations processing, based on inverse problems methodology [8]. This note from the viewpoint of experimental informatibility puts a question on a further research of uniqueness of simultaneous identification as heat properties and boundary conditions.

Mathematical model

We specify the mathematical model (4) and shall consider the measurements circuit with one point of observation. It is required to find such design $\Xi = \{x_1^{opt}, t_i^{opt}\}_{i=1,\dots,4}$ for which known sample of observations

$$u_i^{\delta} = \bar{u}(x_1, t_i) + \varepsilon_i, \quad i = 1, \dots, 4$$

can to define simultaneous a specific heat a_1, a thermal conductivity a_2 and boundary temperatures $v_{0,1}$ with a minimum guaranteed error on *rms*

$$\mu^{(R)} = \min_{x,t} \sqrt{\frac{\mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2}{4}}$$

Here $\mu_{1,2} = (\bar{a}_{1,2} - a_{1,2}) / \bar{a}_{1,2}$ and $\mu_{3,4} = (\bar{v}_{0,1} - v_{0,1}) / \bar{v}_{0,1}$ are relative identification errors.

The solutions of observation fitting equations are shown in Fig.6. The one is expressed the *rms* error dependence from sensor allocation minimizing on observation time.

There are three optimal allocation of a sensor. The global minimum has allocation in the middle of a specimen. Other optimal allocation is near from the specimen boundaries. The variation of observations error doesn't change this allocation.

Thus the using only one internal observation allows to find as specimen properties and its loading factors. This result shows that inverse problems can be considered as a powerful extrapolation tool for experimental data processing. The practical realization of such standpoint is reported in [9].

CONCLUSIONS

Among of observations it is possible to define the most informatibility sample. A small volume of observations can to ensure the identification of a significance number of unknowns. The necessary volume of observation is defined by identifiability conditions. If inverse problem peculiarities are taking into account then it is possible to identify as phenomenological object properties and its loading factors.

The heat properties identification has strictly defined circuit of measurements of a specimen temperatures.

Optimal design must be carried out simultaneous at several direction. Specimen loading, sensors allocation and measurements time are the main characters of design.

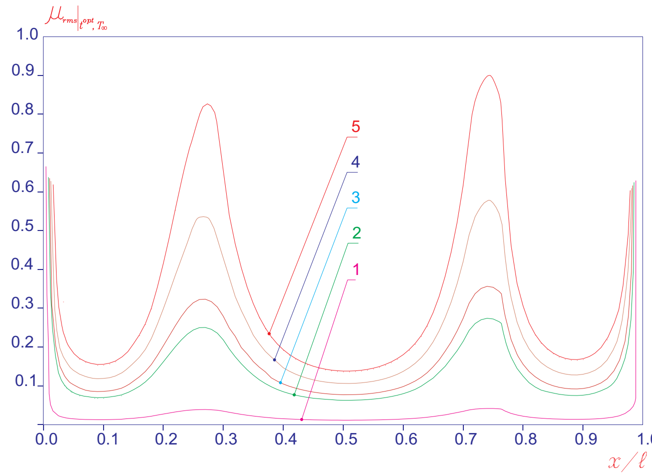

Fig.6. SIMULTANEOUS IDENTIFICATION OF HEAT PROPERTIES AND BOUNDARY TEMPERATURES

1-δ=0.01, 2-δ=0.05, 3–δ=0.06,4–δ=0.08, 5–δ=0.1

A determination of the factors structure of identification errors reduction has a doubtless interest. These factors define the conditions of the identification error decreasing to zero and its dependence mode from sensors allocation.

The analysis shows the existence of statistical indeterminacy of inverse problem solution. The one is described as a corresponding criterion, which has an upper bound of its admissible value.

The further direction of investigation is a generalization on a number of unknowns, nonlinear heat properties identification and other type of boundary conditions.

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