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EXP06

OPTIMIZATION METHOD FOR ONE- AND TWO-DIMENSIONAL INVERSE STEFAN PROBLEMS

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ABSTRACT

In the paper, the one- and two-dimensional two-phase inverse Stefan problems are formulated and described by means of the optimization method. These problems consist of the reconstruction of the function which describes the coefficient of convective heat-transfer, when the position of the moving interface of the phase change is well-known. In numerical calculations the Nelder-Mead optimization method and the alternating phase truncation method were used.

Keywords: Inverse Stefan Problems, Solidification, Alternating Phase Truncation Method, Nelder-Mead Method.

NOMENCLATURE

- a_k Thermal diffusivity in liquid ($k = 1$) or solid phase ($k = 2$) $[m^2/s]$.
- c_k Specific heat $[J/(kg \cdot deg)]$.
- L Latent heat of fusion $[J/kg]$.
- $\boldsymbol{n}$ Unit normal vector to the freezing front.
- T_k Temperature $[K]$.
- T^* Temperature of solidification $[K]$.
- T_∞ Ambient temperature $[K]$.
- $\boldsymbol{v}_n$ The freezing front velocity vector in the normal direction.
- α Coefficient of convective heat-transfer $[W/(m^2 \cdot deg)]$.
- λ_k Thermal conductivity.
- ϱ_k Mass density $[kg/m^3]$.

INTRODUCTION

The inverse problems for differential equations consist of stating the initial conditions, boundary conditions or thermophysical properties of the body. But the insufficiency of input information is compensated by some additional information on the effects of the input conditions. Generally, for the inverse Stefan problem it is assumed that this additional information is the position of the freezing front, its velocity in normal direction or temperature in selected points of the domain.

Most of the papers concerning this field are focused on the one-phase one-dimensional inverse Stefan problems. Papers devoted to two-dimensional problems are not that numerous, part of them have little importance for applications, as regards a way of solution [Colton, 1974, Ang et al., 1997a, Ang et al., 1997b, Bobula and Twardowska, 1985, Stampella and Tarzia, 1989]. Most published materials involve the reconstruction of temperature or heat flux on the boundary of a domain [Ang et al., 1997b, Ang et al., 1998, Grzymkowski and Słota, 1998, Voller, 1992, Zabaras et al., 1988, Zabaras, 1990, Zabaras and Kang, 1993]. In the paper [Vigak, 1989] the distribution of the inner heat sources in a domain is reconstructed. The inverse Stefan problems, where the thermal properties of materials (e.g. thermal conductivity, thermal diffusivity, coefficient of convective heat-transfer etc.) are reconstructed, are discussed in the papers [Stampella and Tarzia, 1989, Tarzia, 1982, Tarzia, 1983, Tarzia, 1991]. Unfortunately, all these papers pertain to semiinfinite domains, but the two-phase problem is considered only in the paper [Stampella and Tarzia, 1989].

In the papers [Gorenflo et al., 1995, Thanh, 1995, Ang et al., 1997a, Ang et al., 1997b] the regularization of inverse Stefan problems is considered. In the paper [Colton, 1974] the solution is found in terms of an infinite series of one-dimensional integrals. Jochum [Jochum, 1982] considers the inverse Stefan problem as a problem of nonlinear approximation theory (see [Jochum, 1980a, Jochum, 1980b]). In the papers [Colton and Reemtsen, 1983, Colton and Reemtsen, 1984] for solutions of one-phase two-dimensional problems authors used a complete family of solutions to the heat equation to minimize the maximal defect in the initial-boundary data. Similar method was used in [Grzymkowski and Słota, 1998] for one-dimensional and in [Grzymkowski and Słota, 1999] for two-dimensional two-phase inverse Stefan problems. The solution, in this method, is found in a linear combination form of the functions satisfing the equation of heat conduction. The coefficients of this combination are determined by the least square method for the boundary of a domain. In the papers [Bénard et al., 1994, Zabaras et al., 1992, Zabaras and Yuan, 1994, Kang and Zabaras, 1995] authors used dynamic programming or minimization techniques in finite- and infinite dimensional space. Unfortunately, the majority of these papers pertain to the one-phase problems, the two-phase problems are considered only in the paper [Ang et al., 1997b, Kang and Zabaras, 1995, Grzymkowski and Słota, 1999, Zabaras et al., 1992, Zabaras and Yuan, 1994].

In this paper, a method for the reconstruction of the function which describes the coefficient of convective heat-transfer is discussed, when the position of the moving interface of the phase change is well-known. The method consists of the minimization of a functional, the value of which is the norm of a difference between given position of the moving interface of the phase change and a position reconstructed from the selected function describing the coefficient of convective heat-transfer. In numerical calculations the Nelder-Mead optimization method [Bunday, 1984] and the alternating phase truncation method [Rogers et al., 1979] were used.

ONE-DIMENSIONAL PROBLEM

Let $\Omega = (a, b) \subset \mathbb{R}$ be a domain. On the boundary of a domain $D = \Omega \times (0, t^*)$ three components are distributed:

$$\Gamma_0 = \{(x, 0);\ x \in [a, b]\}, \tag{1}$$
$$\Gamma_1 = \{(a, t);\ t \in [0, t^*]\}, \tag{2}$$
$$\Gamma_2 = \{(b, t);\ t \in [0, t^*]\}, \tag{3}$$

where initial and boundary conditions are given. Let D_1 (D_2) be this subset of domain D which is occupied by liquid (solid) phase, separated by the freezing front $\Gamma_g = \xi(t)$. We will look for an approximate solution of the following problem:

For given position of freezing front Γ_g, the distribution of temperature T_k in domain D_k ($k = 1, 2$) is calculated as well as function $\alpha(t)$ on boundary Γ_2, what satisfies the following equations (for $k = 1, 2$):

$$\frac{\partial T_k}{\partial t}(x,t) = a_k \frac{1}{x}\frac{\partial}{\partial x}\left(x\,\frac{\partial T_k}{\partial x}(x,t)\right), \quad \text{in } D_k, \tag{4}$$
$$T_1(x,0) = \varphi_0(x), \quad \text{on } \Gamma_0, \tag{5}$$
$$-\lambda_k \frac{\partial T_k}{\partial x}(x,t) = 0, \quad \text{on } \Gamma_1, \tag{6}$$
$$-\lambda_k \frac{\partial T_k}{\partial x}(x,t) = \alpha\big(T_k(x,t) - T_\infty\big), \quad \text{on } \Gamma_2, \tag{7}$$
$$T_k(x,t) = T^*, \quad \text{on } \Gamma_g, \tag{8}$$

$$L\,\varrho_2\,\frac{d\xi}{dt} = -\lambda_1 \frac{\partial T_1(x,t)}{\partial x}\bigg|_{\Gamma_g} + \lambda_2 \frac{\partial T_2(x,t)}{\partial x}\bigg|_{\Gamma_g}. \tag{9}$$

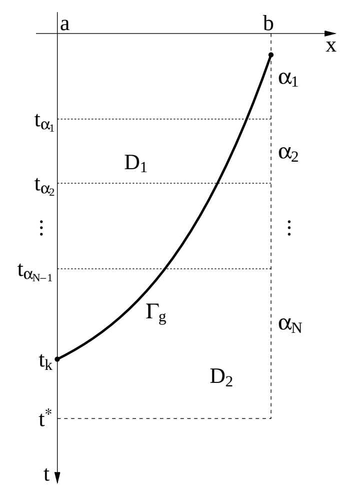

Figure 1. DOMAIN OF THE ONE-DIMENSIONAL PROBLEM.

Let's assume that function ξ^* describing the exact position of the freezing front is well-known. Function $\alpha(t)$ is designated in the form:

$$\alpha(t) = \begin{cases} \alpha_1 & \text{for } t \le t_{\alpha_1}, \\ \alpha_2 & \text{for } t \in (t_{\alpha_1}, t_{\alpha_2}], \\ \vdots & \\ \alpha_{N-1} & \text{for } t \in (t_{\alpha_{N-2}}, t_{\alpha_{N-1}}], \\ \alpha_N & \text{for } t > t_{\alpha_{N-1}}. \end{cases} \tag{10}$$

Let V_α denote the set of all functions in form (10), where $\alpha_i \in \mathbb{R}$.

For fixed function $\alpha \in V_\alpha$ problem in (4)–(9) the direct Stefan problem occurs and its solution makes it possible to find the position of the freezing front corresponding to function $\alpha(t)$. For solving the direct Stefan problem we used the alternating phase truncation method. The minimized functional can be represented as:

$$J(\alpha) = \left(\sum_{i=1}^{M} (\xi_i - \xi_i^*)^2 \right)^{1/2}, \tag{11}$$

where ξ is the position of the freezing front corresponding to function $\alpha(t)$. We will look for such element $\alpha_m \in V_\alpha$, that:

$$J(\alpha_m) = \inf_{\alpha \in V_\alpha} J(\alpha). \tag{12}$$

To look for the minimum of functional J we used the Nelder-Mead optimization method.

In real processes the function $\alpha(t)$ describing the coefficient of convective heat-transfer, doesn't have an arbitrary value. Therefore, the problem of minimization with constraints has some practical application. Assuming that:

$$V_\alpha^0 = \left\{ \alpha \in V_\alpha, \alpha_i \geq 0 \right\}, \tag{13}$$

$$V_\alpha^p = \left\{ \alpha \in V_\alpha, \alpha_i \in [p_{1i}, p_{2i}] \right\}. \tag{14}$$

In the problem with constraints we will look for $\alpha_m \in V$, where $V = V_\alpha^0$ or $V = V_\alpha^p$, that:

$$J(\alpha_m) = \inf_{\alpha \in V} J(\alpha). \tag{15}$$

Numerical example

Exemplary solution of the one-dimensional two-phase inverse Stefan problem is presented, where: $\Omega = (0, 0.08)$, $N = 3$, $a_k = \lambda_k/(c_k\,\varrho_k)$ for $k = 1, 2$, $\lambda_1 = 33$, $\lambda_2 = 30$, $c_1 = 800$, $c_2 = 690$, $\varrho_1 = 7000$, $\varrho_2 = 7500$, $L = 270000$. The temperature of solidification is $T^* = 1773$, ambient temperature is $T_\infty = 323$ and initial temperature is equal to $\varphi_0(x) = 1813$. In all presented examples the exact value of function $\alpha(t)$ is equal to:

$$\alpha(t) = \begin{cases} 1200 & \text{for } t \leq t_{\alpha_1}, \\ 800 & \text{for } t \in (t_{\alpha_1}, t_{\alpha_2}], \\ 250 & \text{for } t > t_{\alpha_2}, \end{cases} \tag{16}$$

where $t_1 = 38$ and $t_2 = 93$.

Table 1. RESULTS OF THE CALCULATIONS FOR DIFFERENT STARTING POINTS (α_s – STARTING POINT, α_m – FOUND POINT OF MINIMUM).

α_s			α_m		
α_{s1}	α_{s2}	α_{s3}	α_{m1}	α_{m2}	α_{m3}
1000	500	500	1213	787	252
1000	700	200	1200	802	249
900	900	600	1200	801	250
850	650	350	1157	844	246
800	800	800	1198	801	249
750	600	450	1193	810	248
700	300	450	1200	802	248
0	0	0	1201	799	251

Table 2. RESULTS OF THE CALCULATIONS FOR 40 CONTROL POINTS.

Per.	V_α^0			V_α^p		
	α_{m1}	α_{m2}	α_{m3}	α_{m1}	α_{m2}	α_{m3}
0%	1213	787	252	1201	798	251
2.5%	1208	752	267	1276	681	275
5%	1135	822	263	1260	697	274
7.5%	1210	665	298	1174	691	301
10%	1321	796	210	1313	748	239
12.5%	1480	499	303	1317	712	253
15%	1453	887	161	1452	877	158
20%	1581	683	223	1370	911	177
25%	1394	373	360	1208	615	304

Table 1 describes results of the calculations of problem (15) in V_α^0 for different starting point. In the calculations 40 control points ($M = 40$) were used.

Tables 2–4 include calculating minimal point α_m. In the calculations of functional (11) 40, 10 or 5 control points of the perturbation position of the freezing front were used (with maximal error ranging from 0% to 25%). Minimum was designated in set V_α^0 and set V_α^p, where $\alpha_1 \in [1000, 500]$, $\alpha_2 \in [500, 1000]$, $\alpha_3 \in [0, 500]$. In these examples calculations started with initial value $(\alpha_1, \alpha_2, \alpha_3) = (1000, 500, 500)$.

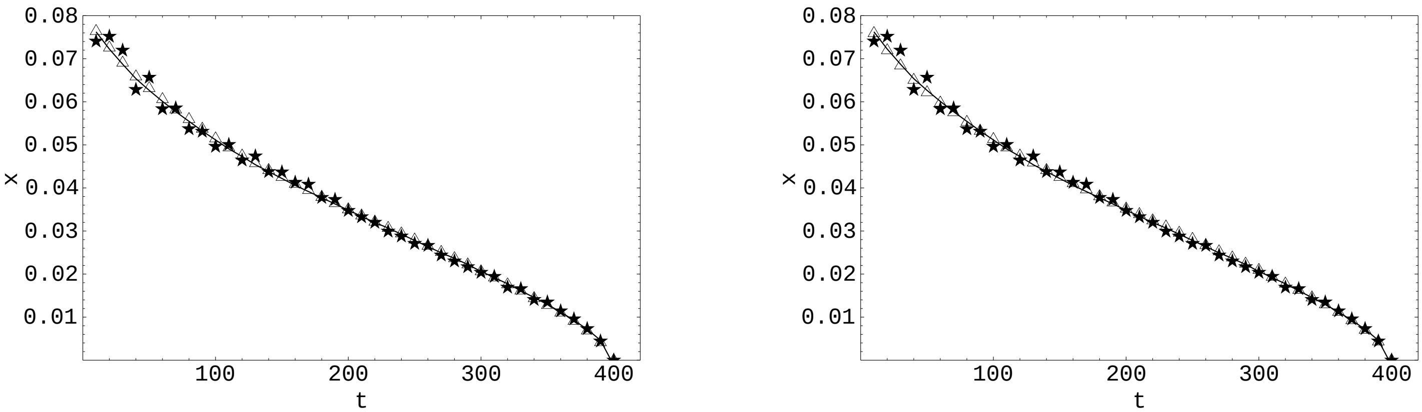

Figure 2. RESULTS OF THE CALCULATIONS IN THE CASE $M = 40$ AND PERTURBATION EQUALS 5% (MINIMUM FOUND IN SET V_α^0 (LEFT FIGURE) OR V_α^p (RIGHT FIGURE), SOLID LINE – EXACT POSITION, $\star$ – ASSUMED PERTURBATION POSITION (CONTROL POINTS), $\triangle$ – RECONSTRUCTED POSITION).

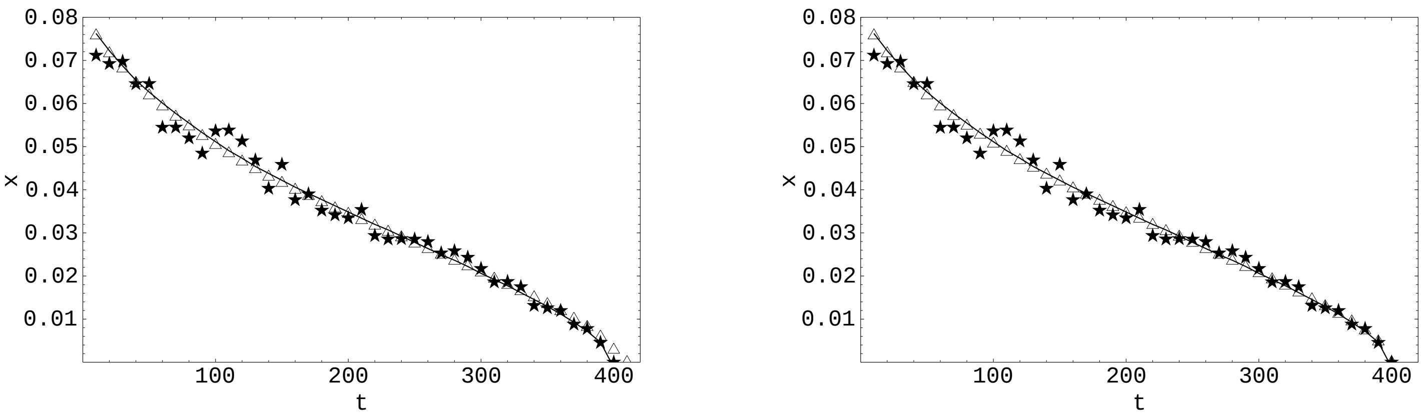

Figure 3. RESULTS OF THE CALCULATIONS IN THE CASE $M = 40$ AND PERTURBATION EQUALS 10% (MINIMUM FOUND IN SET V_α^0 (LEFT FIGURE) OR V_α^p (RIGHT FIGURE), SOLID LINE – EXACT POSITION, $\star$ – ASSUMED PERTURBATION POSITION (CONTROL POINTS), $\triangle$ – RECONSTRUCTED POSITION).

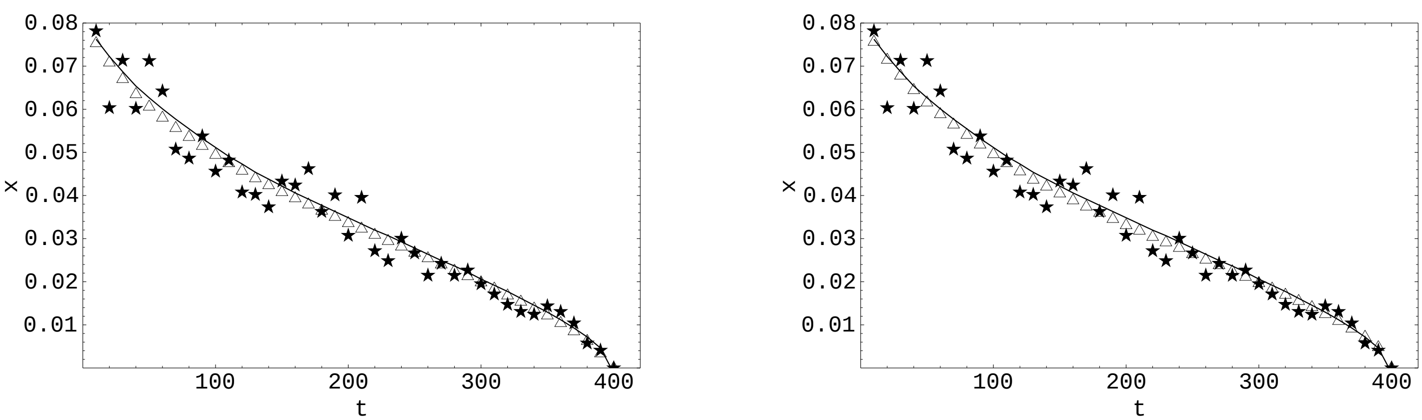

Figure 4. RESULTS OF THE CALCULATIONS IN THE CASE $M = 40$ AND PERTURBATION EQUALS 20% (MINIMUM FOUND IN SET V_α^0 (LEFT FIGURE) OR V_α^p (RIGHT FIGURE), SOLID LINE – EXACT POSITION, $\star$ – ASSUMED PERTURBATION POSITION (CONTROL POINTS), $\triangle$ – RECONSTRUCTED POSITION).

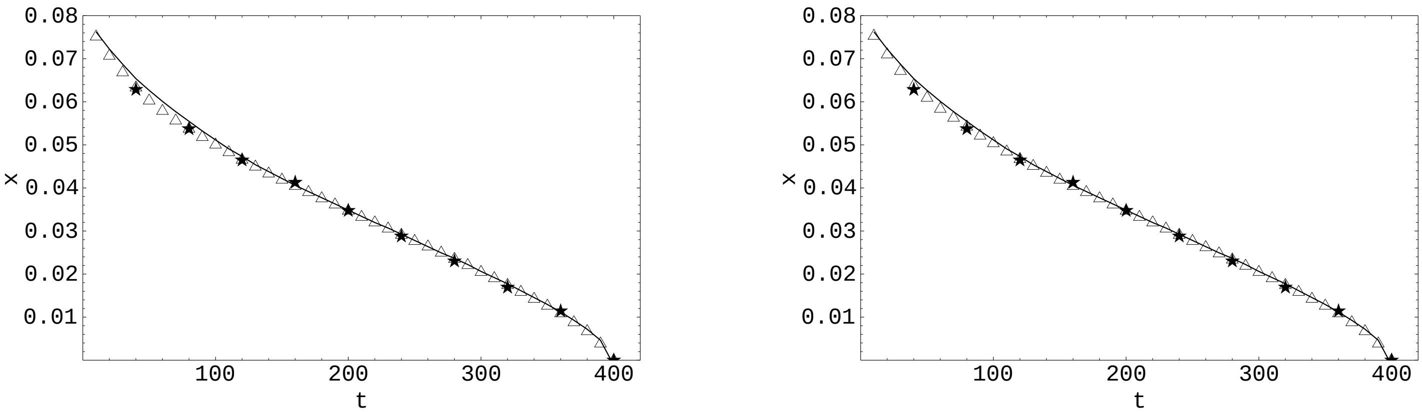

Figure 5. RESULTS OF THE CALCULATIONS IN THE CASE $M = 10$ AND PERTURBATION EQUALS 5% (MINIMUM FOUND IN SET V_α^0 (LEFT FIGURE) OR V_α^p (RIGHT FIGURE), SOLID LINE – EXACT POSITION, $\star$ – ASSUMED PERTURBATION POSITION (CONTROL POINTS), $\triangle$ – RECONSTRUCTED POSITION).

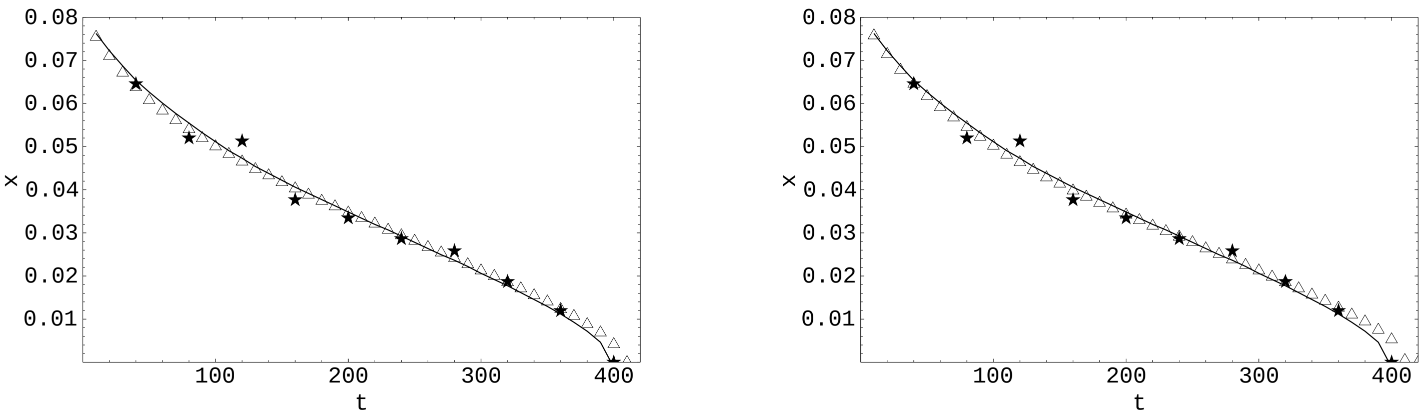

Figure 6. RESULTS OF THE CALCULATIONS IN THE CASE $M = 10$ AND PERTURBATION EQUALS 10% (MINIMUM FOUND IN SET V_α^0 (LEFT FIGURE) OR V_α^p (RIGHT FIGURE), SOLID LINE – EXACT POSITION, $\star$ – ASSUMED PERTURBATION POSITION (CONTROL POINTS), $\triangle$ – RECONSTRUCTED POSITION).

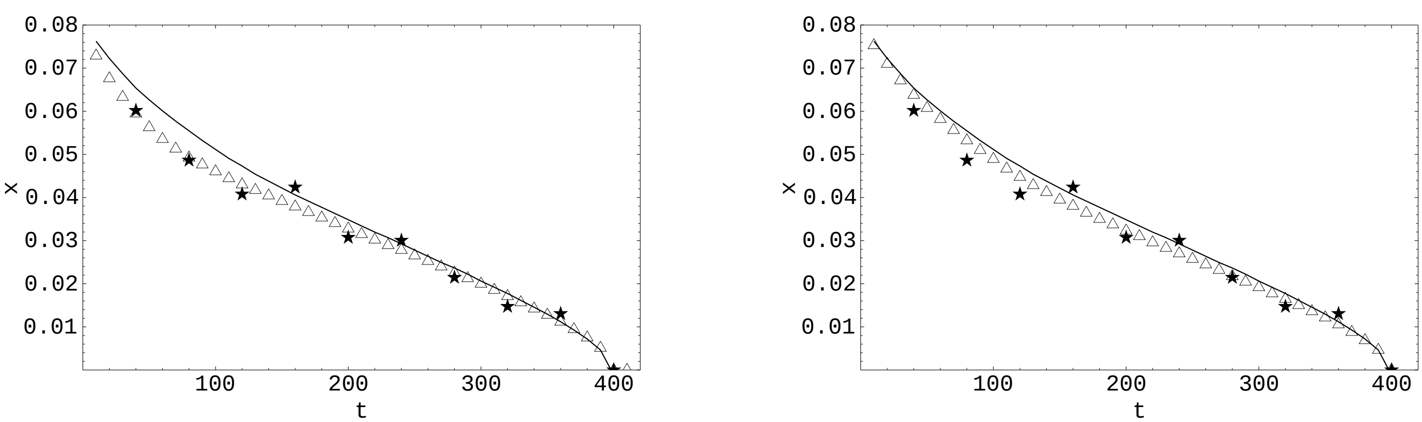

Figure 7. RESULTS OF THE CALCULATIONS IN THE CASE $M = 10$ AND PERTURBATION EQUALS 20% (MINIMUM FOUND IN SET V_α^0 (LEFT FIGURE) OR V_α^p (RIGHT FIGURE), SOLID LINE – EXACT POSITION, $\star$ – ASSUMED PERTURBATION POSITION (CONTROL POINTS), $\triangle$ – RECONSTRUCTED POSITION).

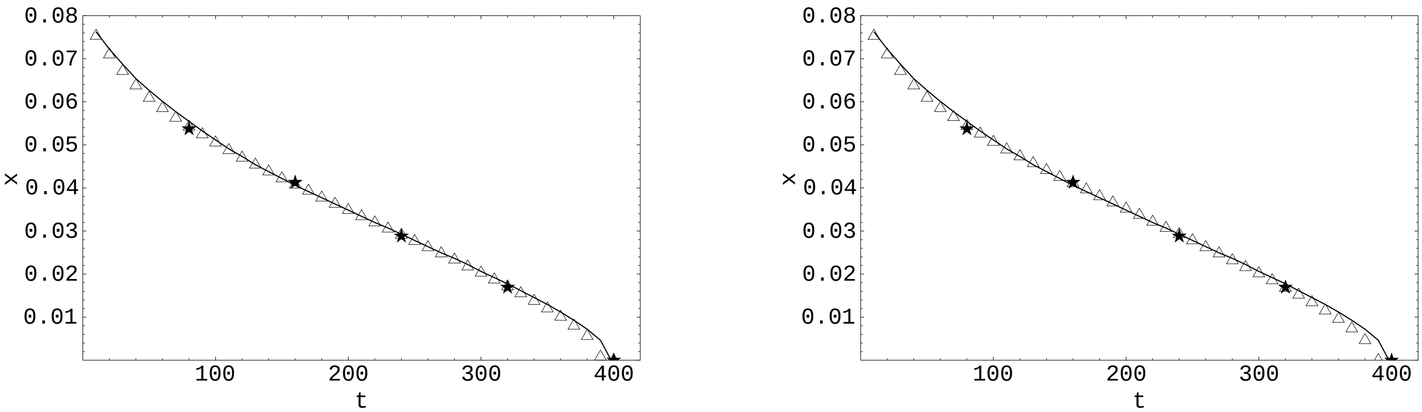

Figure 8. RESULTS OF THE CALCULATIONS IN THE CASE $M = 5$ AND PERTURBATION EQUALS 5% (MINIMUM FOUND IN SET V_α^0 (LEFT FIGURE) OR V_α^p (RIGHT FIGURE), SOLID LINE – EXACT POSITION, $\star$ – ASSUMED PERTURBATION POSITION (CONTROL POINTS), $\triangle$ – RECONSTRUCTED POSITION).

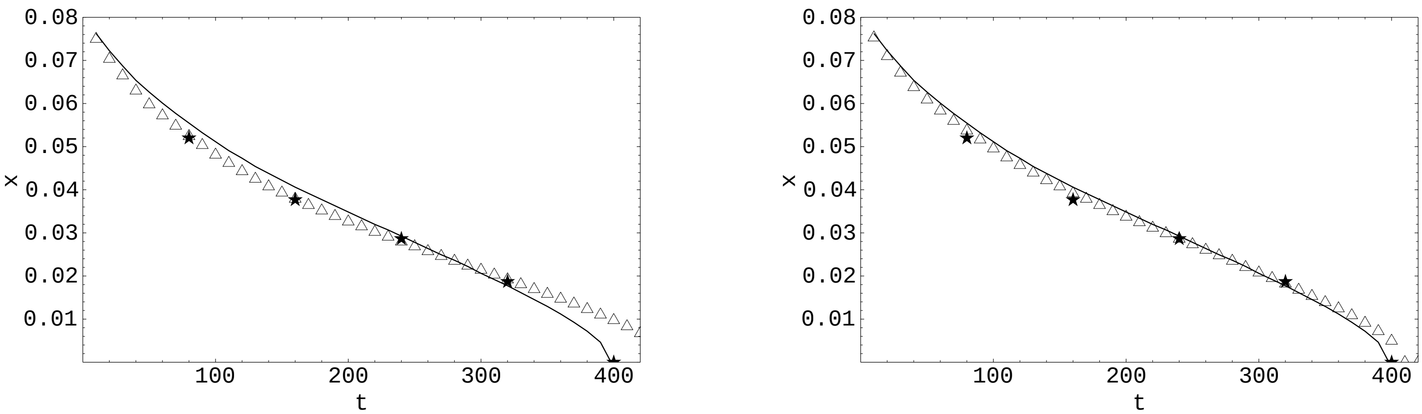

Figure 9. RESULTS OF THE CALCULATIONS IN THE CASE $M = 5$ AND PERTURBATION EQUALS 10% (MINIMUM FOUND IN SET V_α^0 (LEFT FIGURE) OR V_α^p (RIGHT FIGURE), SOLID LINE – EXACT POSITION, $\star$ – ASSUMED PERTURBATION POSITION (CONTROL POINTS), $\triangle$ – RECONSTRUCTED POSITION).

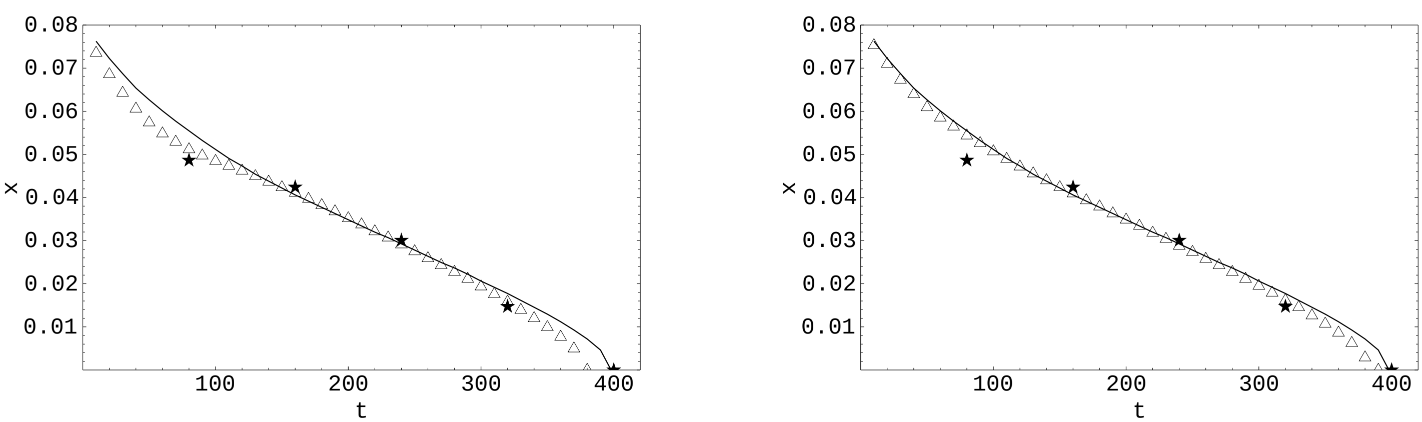

Figure 10. RESULTS OF THE CALCULATIONS IN THE CASE $M = 5$ AND PERTURBATION EQUALS 20% (MINIMUM FOUND IN SET V_α^0 (LEFT FIGURE) OR V_α^p (RIGHT FIGURE), SOLID LINE – EXACT POSITION, $\star$ – ASSUMED PERTURBATION POSITION (CONTROL POINTS), $\triangle$ – RECONSTRUCTED POSITION).

Table 3. RESULTS OF THE CALCULATIONS FOR 10 CONTROL POINTS.

Per.	V_α^0			V_α^p		
	α_{m1}	α_{m2}	α_{m3}	α_{m1}	α_{m2}	α_{m3}
0%	1325	651	284	1269	725	260
2.5%	1023	915	257	1105	782	293
5%	1631	518	268	1496	593	264
7.5%	1276	777	224	1296	736	234
10%	1530	613	220	1354	796	185
12.5%	1244	697	251	1198	689	267
15%	766	1758	53	1386	850	154
20%	2964	149	203	1497	927	151
25%	797	298	666	1005	505	425

Figures 2–10 describe results of reconstruction of the freezing front for calculated minimal point α_m. In the calculations of functional (11) 40, 10 or 5 control points of the perturbation position of the freezing front were used (with maximal error equal 5%, 10% or 20%). Left figures describe reconstructed position of the freezing front for minimization problem in set V_α^0 and right figures for minimization in set V_α^p.

Table 4. RESULTS OF THE CALCULATIONS FOR 5 CONTROL POINTS.

Per.	V_α^0			V_α^p		
	α_{m1}	α_{m2}	α_{m3}	α_{m1}	α_{m2}	α_{m3}
0%	1319	666	274	1319	666	274
2.5%	1073	860	275	1082	842	279
5%	1500	548	297	1500	505	327
7.5%	2272	25	377	1406	695	184
10%	1718	825	74	1496	761	170
12.5%	2427	10	290	1329	756	150
15%	366	2327	25	1129	995	175
20%	2464	14	423	1482	533	338
25%	45	735	825	1015	502	486

TWO-DIMENSIONAL PROBLEM

Let $\Omega = (c_1, c_2) \times (d_1, d_2) \subset \mathbb{R}^2$ be a domain. On the boundary of a domain $D = \Omega \times (0, t^*)$ five components are distributed:

$$\Gamma_0 = \{(x, y, 0);\ x \in [c_1, c_2],\ y \in [d_1, d_2]\}, \tag{17}$$
$$\Gamma_1 = \{(c_1, y, t);\ y \in [d_1, d_2],\ t \in [0, t^*]\}, \tag{18}$$
$$\Gamma_2 = \{(x, d_1, t);\ x \in [c_1, c_2],\ t \in [0, t^*]\}, \tag{19}$$
$$\Gamma_3 = \{(c_2, y, t);\ y \in [d_1, d_2],\ t \in [0, t^*]\}, \tag{20}$$
$$\Gamma_4 = \{(x, d_2, t);\ x \in [c_1, c_2],\ t \in [0, t^*]\}, \tag{21}$$

where initial and boundary conditions are given. Let D_1 (D_2) be this subset of domain D which is occupied by liquid (solid) phase, separated by the freezing front Γ_g. We will look for an approximate solution of the following problem:

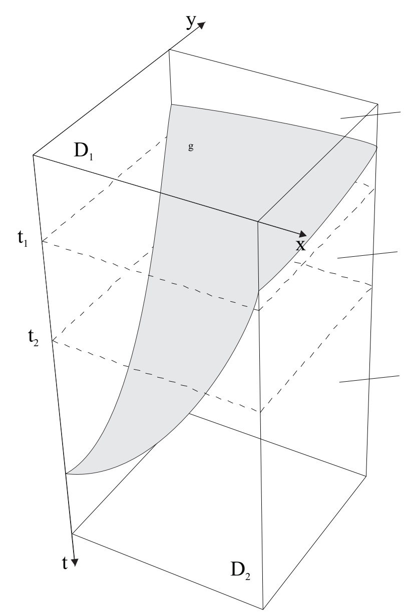

Figure 11. DOMAIN OF THE TWO-DIMENSIONAL PROBLEM.

For given position of freezing front Γ_g, the distribution of temperature T_k in domain D_k $(k = 1, 2)$ is calculated as well as function $\alpha(x, y, t)$ on boundary $\Gamma_3 \cup \Gamma_4$, what satisfies the following equations (for $k = 1, 2$):

$$\frac{\partial T_k}{\partial t}(x, y, t) = a_k \nabla^2 T_k(x, y, t), \quad \text{in } D_k, \tag{22}$$
$$T_1(x, y, 0) = \varphi_0(x, y), \qquad \text{on } \Gamma_0, \tag{23}$$
$$-\lambda_k \frac{\partial T_k}{\partial \boldsymbol{n}}(x, y, t) = 0, \qquad \text{on } \Gamma_1 \cup \Gamma_2, \tag{24}$$

$$-\lambda_k \frac{\partial T_k}{\partial \boldsymbol{n}}(x,y,t) = \alpha\big(T_k(x,y,t) - T_\infty\big), \text{ on } \Gamma_3 \cup \Gamma_4, \quad (25)$$

$$T_k(x,y,t) = T^*, \qquad \text{on } \Gamma_g, \qquad (26)$$

$$-\lambda_1 \frac{\partial T_1(x,y,t)}{\partial \boldsymbol{n}}\bigg|_{\Gamma_g} + \lambda_2 \frac{\partial T_2(x,y,t)}{\partial \boldsymbol{n}}\bigg|_{\Gamma_g} = L\,\varrho_2\,\boldsymbol{v}_n. \qquad (27)$$

Let's assume that function ξ^* ($y = \xi^*(x,t)$ or $x = \xi^*(y,t)$) describing the exact position of the freezing front is well-known. Function $\alpha(x,y,t)$ is designated in the form:

$$\alpha(x,y,t) = \begin{cases} \alpha_1 & \text{for } t \le t_1 \wedge y = d_2, \\ \alpha_3 & \text{for } t \in (t_1, t_3] \wedge y = d_2, \\ \vdots & \\ \alpha_{2N_1-1} & \text{for } t > t_{2N_1-3} \wedge y = d_2, \\ \alpha_2 & \text{for } t \le t_2 \wedge x = c_2, \\ \alpha_4 & \text{for } t \in (t_2, t_4] \wedge x = c_2, \\ \vdots & \\ \alpha_{2N_2} & \text{for } t > t_{2N_2-2} \wedge x = c_2. \end{cases} \qquad (28)$$

For fixed function $\alpha \in V_\alpha^0$ problem in (22)–(27) the direct Stefan problem occurs and its solution makes it possible to find (ξ) the position of the freezing front corresponding to function $\alpha(x,y,t)$. For solving the direct Stefan problem we used the alternating phase truncation method [Rogers et al., 1979]. The minimized functional can be represented as:

$$J(\alpha) = \left(\sum_{j=1}^{M_1} \sum_{i=1}^{M_2} \left(\xi_{ji} - \xi_{ji}^*\right)^2 \right)^{1/2}. \qquad (29)$$

Let V_α denote the set of all functions in form (28), where $\alpha_i \in \mathbb{R}$. Assuming that:

$$V_\alpha^0 = \Big\{ \alpha \in V_\alpha, \alpha_i \ge 0 \Big\}. \qquad (30)$$

We will look for such element $\alpha_m \in V_\alpha^0$, that:

$$J(\alpha_m) = \inf_{\alpha \in V_\alpha^0} J(\alpha). \qquad (31)$$

To look for the minimum of functional J we used the Nelder-Mead optimization method [Bunday, 1984].

NUMERICAL EXAMPLE

Exemplary solution of the two-dimensional two-phase inverse Stefan problem is presented, where: $\Omega = (0, 0.08) \times (0, 0.08)$, $N_1 = N_2 = 3$, $a_k = \lambda_k/(c_k\,\varrho_k)$ for $k = 1, 2$, $\lambda_1 = 33$, $\lambda_2 = 30$, $c_1 = 800$, $c_2 = 690$, $\varrho_1 = 7000$, $\varrho_2 = 7500$, $L = 270000$, $\varrho = 7500$. The temperature of solidification is $T^* = 1773$, ambient temperature is $T_\infty = 323$ and initial temperature is equal to $\varphi_0(x) = 1813$.

We known positions of freezing front in three directions ($M_1 = 3$): parallel to axis Ox ($j = 1$), parallel to axis Oy ($j = 2$) and parallel to diagonal of domain Ω ($j = 3$). The calculations are conducted for the exact and perturbation position of the freezing front (with maximal error equal to 2.5% or 5%). The exact positions of the freezing front and the approximated value reconstructed for the found function $\alpha_m(x,y,t)$ are presented in Figures 12–14 for 520, 52 or 14 control points ($M_2 = 520$, $M_2 = 52$, $M_2 = 14$), respectively.

In all presented examples, the calculations started with initial value $(\alpha_1, \alpha_3, \alpha_5, \alpha_2, \alpha_4, \alpha_6) = (1000, 500, 500, 1000, 500, 500)$. The exact value of function α is equal to:

$$\alpha(x,y,t) = \begin{cases} 800 & \text{for } t \le t_1 \wedge y = d_2, \\ 500 & \text{for } t \in (t_1, t_2] \wedge y = d_2, \\ 200 & \text{for } t > t_2 \wedge y = d_2, \\ 1200 & \text{for } t \le t_1 \wedge x = c_2, \\ 800 & \text{for } t \in (t_1, t_2] \wedge x = c_2, \\ 250 & \text{for } t > t_2 \wedge x = c_2, \end{cases} \qquad (32)$$

where $t_1 = 38$, $t_2 = 93$, and $c_2 = d_2 = 0.08$. Table 5 describes the results of calculations by optimization method of the function α_m which describes the coefficient of convective heat-transfer.

CONCLUSION

The obtained results confirm the usability of this method for solving the inverse Stefan problem. For exact values, the optimization method renders exact results. The method may be successfully used in design problems, in which it's necessary to select the boundary conditions so that the freezing front takes the given position (for example in continuous casting of steel). In the optimization method in spite of a certain difference between exact and reconstructed values of function α, the position of the freezing front is reconstructed very well every time (especially in case of looking for minimum in set V_α^p or for greater number of control points). The obtained results show that for correct calculations of minimal point α_m it is enough to increase a number of control points or look for minimum in set V_α^p with appropriate choice of intervals $[p_{1i}, p_{2i}]$.

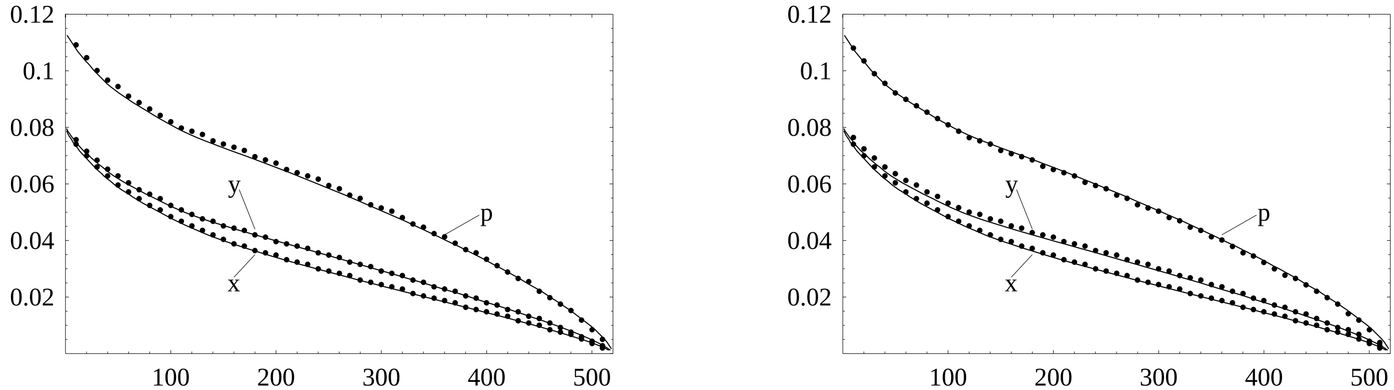

Figure 12. RESULTS OF THE CALCULATIONS FOR $M_2 = 520$ (LEFT FIGURE – FOR PERTURBATION 2.5 %, RIGHT ONES – FOR PERTURBATION 5 %, SOLID LINE – EXACT VALUE, DASHED LINE – RECONSTRUCTED VALUE, x – POSITION OF THE FREEZING FRONT IN DIRECTION OF PARALLEL TO AXIS Ox, y – POSITION OF THE FREEZING FRONT IN DIRECTION OF PARALLEL TO AXIS Oy, p – POSITION OF THE FREEZING FRONT IN DIRECTION OF PARALLEL TO DIAGONAL OF DOMAIN Ω).

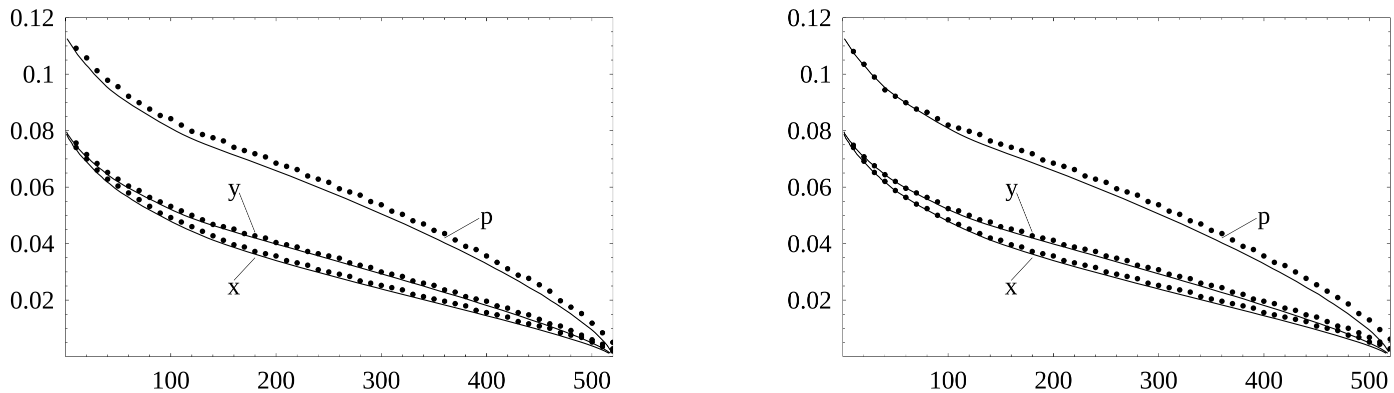

Figure 13. RESULTS OF THE CALCULATIONS FOR $M_2 = 52$ (LEFT FIGURE – FOR PERTURBATION 2.5 %, RIGHT ONES – FOR PERTURBATION 5 %, SOLID LINE – EXACT VALUE, DASHED LINE – RECONSTRUCTED VALUE, x – POSITION OF THE FREEZING FRONT IN DIRECTION OF PARALLEL TO AXIS Ox, y – POSITION OF THE FREEZING FRONT IN DIRECTION OF PARALLEL TO AXIS Oy, p – POSITION OF THE FREEZING FRONT IN DIRECTION OF PARALLEL TO DIAGONAL OF DOMAIN Ω).

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Table 5. RESULTS OF THE CALCULATIONS OF FUNCTION α_m.

	exact	$M_2 = 520$			$M_2 = 52$			$M_2 = 14$		
		0%	2.5%	5%	0%	2.5%	5%	0%	2.5%	5%
α_1	800	803	762	873	773	700	934	773	700	934
α_3	500	508	513	545	444	472	336	444	472	336
α_5	200	202	204	201	203	206	202	198	200	225
α_2	1200	1198	1197	1041	1125	1150	1361	1125	1150	1361
α_4	800	876	884	870	866	852	652	866	852	652
α_6	250	254	252	255	251	249	254	258	259	226

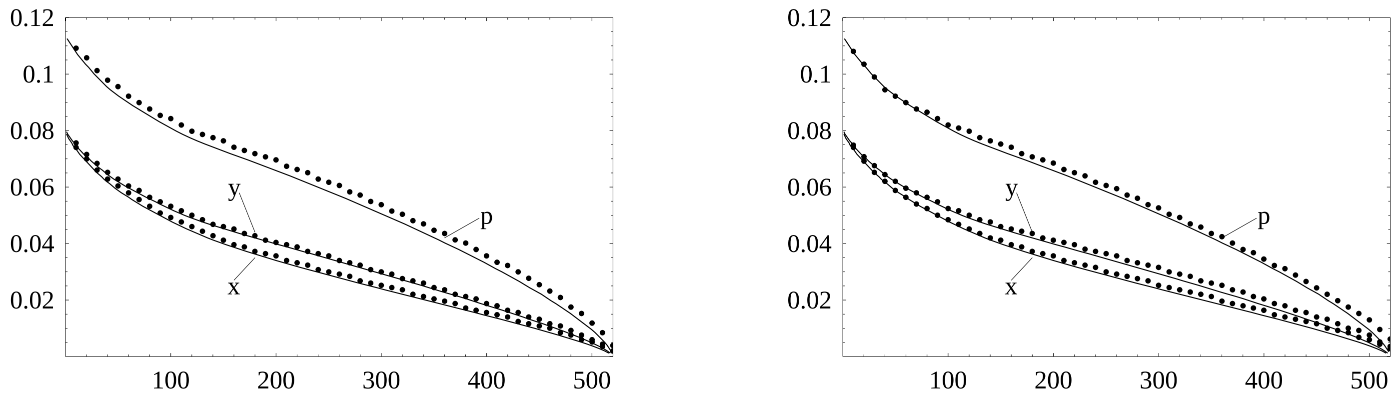

Figure 14. RESULTS OF THE CALCULATIONS FOR $M_2 = 14$ (LEFT FIGURE – FOR PERTURBATION 2.5 %, RIGHT ONES – FOR PERTURBATION 5 %, SOLID LINE – EXACT VALUE, DASHED LINE – RECONSTRUCTED VALUE, x – POSITION OF THE FREEZING FRONT IN DIRECTION OF PARALLEL TO AXIS Ox, y – POSITION OF THE FREEZING FRONT IN DIRECTION OF PARALLEL TO AXIS Oy, p – POSITION OF THE FREEZING FRONT IN DIRECTION OF PARALLEL TO DIAGONAL OF DOMAIN Ω).

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