

NUMERICAL SOLUTION OF GENERALIZED 2-D IHCP BY DISCRETE MOLLIFICATION

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ABSTRACT

We present a proof of stability and convergence of an automatic numerical space marching algorithm, based on discrete mollification and generalized cross validation, for the numerical solution of the generalized 2-D IHCP.

KEY WORDS

Generalized 2-D IHCP, Mollification, Finite Differences,
 Automatic Filtering

\tilde{z}_{ij}^n	Computed 2nd derivative in y direction
$C; C_{\pm}; C_1; C_2$	Generic constants
$G; G^2$	Discrete functions
$I; I_{\pm}$	Domain intervals
J_{\pm}	Mollification operator
$\pm = (\pm_1; \pm_2)$	Mollification radii
$j \pm j_1$	Maximum of \pm_1 and \pm_2
$j \pm j_1 - 1$	Minimum of \pm_1 and \pm_2
l^2	Discrete level
Δx	Space increment
C_i	Maximum of $(r_{ij}^n; w_{ij}^n; q_{ij}^n)$

NOTATION

D	Discrete centered difference operator
K	Discrete interval
$M; N$	Dimension parameters
$h; l; k$	Space and time steps
p	Mollification parameter
t	Time coordinate
u	Exact temperature
v	Regularized temperature
x	Space coordinates
g_j	Discrete components
q_{ij}^n	Computed time heat flux derivative
q_{ij}^n	Computed time temperature derivative
r_{ij}^n	Computed temperature
u_x	Exact heat flux
v_x	Regularized heat flux
w_{ij}^n	Computed heat flux

1. INTRODUCTION

In this paper, the 2-D generalized heat conduction problem is investigated and an automatic space marching algorithm, based on \pm_1 mollification and the generalized cross validation (CV) procedure, is presented and analyzed. This method does not require any information about the initial temperature distribution and the amount and/or characteristics of the noise in the data. Moreover, the mollification parameters are chosen automatically.

The necessary theoretical and computational background for combining mollification and CV has been developed in a sequence of previous papers and the reader is urged to consult the corresponding references for completeness. Zhan and Murio (1998a) presented a numerical space marching scheme for the identification of parameters in the one-dimensional IHCP. The topic of numerical dif-

ferentiation of noisy data is discussed in detail in M. Uorio et al. (1998). Surface fitting and the numerical approximation of 2-D gradient fields have been extensively investigated in Z. Han and M. Uorio (1998b).

This paper is organized as follows: a short description of pertinent notation and preliminary results is presented in section 2. In section 3, the generalized 2-D IH CP is discussed. This section includes the space marching scheme, the proof of stability and convergence of the algorithm, and some numerical examples.

2. PRELIMINARY RESULTS

This section discusses the main results on stable numerical computation of 2-D gradients by the mollification method. We introduce the following notation. Let $x = (x_1; x_2)$; $p = (p_1; p_2)$; and $\pm = (\pm_1; \pm_2)$; where $p_i > 0$; $\pm_i > 0$; $x \in 2\mathbb{R}^2$ and consider the sets $I = [0; 1] \times [0; 1]$ and $K = f(x_1; x_2) : 1 \leq i \leq m; 1 \leq j \leq n; g \geq 1$, with $0 \leq x_1^{(i)} < x_1^{(i+1)} < \dots < x_1^{(m)}$, $1 \leq i \leq m$; $0 \leq x_2^{(j)} < x_2^{(j+1)} < \dots < x_2^{(n)}$, $1 \leq j \leq n$. We also assume that

$$\begin{aligned} x_1^{(i)} - x_1^{(i+1)} &= x_1^{(i+1)} - x_1^{(i)} - \Delta x_1; \quad i = 2, \dots, m-1; \\ x_2^{(j)} - x_2^{(j+1)} &= x_2^{(j+1)} - x_2^{(j)} - \Delta x_2; \quad j = 2, \dots, n-1; \end{aligned}$$

The function G^2 is the given perturbed discrete version of g . In order to approximate $r(g)$ computations are carried out by using the centered differences of $J_{\pm}G^2$. That is, $D(J_{\pm}G^2)$ is used to approximate $r(J_{\pm}G^2)$ in I_{\pm} . Here $D = (D_{x_1}, D_{x_2})$, D_{x_i} ($i = 1, 2$) denotes the centered difference operator with respect to the variable x_i , and

$$I_{\pm} = [p_1 \pm_1 + 4x_1; 1; p_1 \pm_1 + 4x_1] \times [p_2 \pm_2 + 4x_2; 1; p_2 \pm_2 + 4x_2].$$

We can now state the necessary results needed to prove the stability and convergence of the space marching algorithm used in solving the generalized 2-D IH CP presented in section 3. The proofs of these statements can be found in the indicated literature.

PROPOSITION 2.1. Let $G = fg_j = g(x_1^{(i)}; x_2^{(j)}) : 1 \leq i \leq m; 1 \leq j \leq n$ be the discrete version of g ; $G = fg_j = g_j + \frac{\partial^2 g}{\partial x_1^2} j^2 + \frac{\partial^2 g}{\partial x_2^2} j^2 + \dots + \frac{\partial^2 g}{\partial x_1 \partial x_2} j \cdot 2; 1 \leq i \leq m; 1 \leq j \leq n$; and $rg2C^{0,1}(I) \in C^{0,1}(I)$; If G, G^2 satisfy $KG \in G^2K \in \mathbb{K}$, then

$$jjD(J_{\pm}G^2)_i r(g) jj_{1,\epsilon} \leq C \frac{\mu}{j \pm j_1} + \frac{\Delta x}{j \pm j_1} + \frac{\Delta x^2}{j \pm j_1} + C_{\pm}(\Delta x)^2;$$

and

$$jjD(J_{\pm}G^2)_i r(J_{\pm}G) jj_{1,\epsilon} \leq \frac{C}{j \pm j_1} (\mu + \Delta x) + C_{\pm}(\Delta x)^2;$$

Lemmas 3.7 and 3.8 in Z. Han and M. Uorio (1998b) prove this proposition.

Let G be a discrete function on K . We define $D_{\pm}^{\pm}G = D(J_{\pm}G)_{jk}$. The next theorem states that D_{\pm}^{\pm} is a bounded operator:

THEOREM 2.2. There exists a constant C such that

$$jjD_{\pm}^{\pm}G jj_{1,K \setminus \{k\}} \leq \frac{C}{j \pm j_1} jjG jj_{1,K};$$

The proof of this theorem can be found in Z. Han and M. Uorio (1998b).

THEOREM 2.3. If $g2C^{0,1}(I)$ and G is a discrete version of g , then there exists a constant C such that for $i = 1, 2; \dots, m; j = 1, 2, \dots, n$,

$$\left| \frac{\partial^2}{\partial x_a^2} (J_{\pm}G)(x_1^{(i)}; x_2^{(j)}) \right| \leq \left| \frac{\partial^2}{\partial x_a^2} (J_{\pm}G^2)(x_1^{(i)}; x_2^{(j)}) \right| \leq C \frac{\Delta x}{j \pm j_1^2};$$

$a = 1, 2$. Here $D_{x_a}^2(f)(x_1; x_2)$ denotes the centered difference approximation of $\frac{\partial^2 f}{\partial x_a^2}(x_1; x_2)$ with grid size Δx_a .

THEOREM 2.4. Let G and G^2 be discrete functions defined on K ; satisfying $jjG jj_{1,K} \leq C$. Then for $i = 1, 2; \dots, m; j = 1, 2, \dots, n$,

$$\left| \frac{\partial^2}{\partial x_a^2} (J_{\pm}G)(x_1^{(i)}; x_2^{(j)}) \right| \leq \left| \frac{\partial^2}{\partial x_a^2} (J_{\pm}G^2)(x_1^{(i)}; x_2^{(j)}) \right| \leq C \frac{\Delta x^2}{j \pm j_1^2};$$

$a = 1, 2$.

Theorems 2.3 and 2.4 are natural extensions of the previous theory to the second order derivative functions in the 2-dimensional case.

3. THE GENERALIZED 2-D IH CP

A space marching algorithm solving the generalized 2-dimensional inverse heat conduction problem is presented in this section.

We consider the problem of determining $u(x; y; t)$, $u_x(x; y; t)$, $u_y(x; y; t)$ and $u_t(x; y; t)$ satisfying

$$\begin{aligned} u_t &= u_{xx} + u_{yy}; & x > 0; 0 < y < 1; 0 < t < 1; \\ u(0; y; t) &= u_l(y; t); & 0 \leq y \leq 1; 0 \leq t \leq 1; \\ u_x(0; y; t) &= u_l(y; t); & 0 \leq y \leq 1; 0 \leq t \leq 1; \end{aligned}$$

The available data u_0^2 and u_1^2 for u_0 and u_1 respectively, are discrete noisy functions with maximum noise level σ^2 . We assume that the data are given on the discrete set $f(y_j; t_n) : y_j = j \cdot h; t_n = nh; 1 \leq j \leq M; 1 \leq n \leq N$ with $h = 1/M$ and $N = 1/h$.

The regularized problem based on mdlli⁻cation is formulated as follows: determine $v(x; y; t)$, $v_x(x; y; t)$, $v_y(x; y; t)$ and $v_t(x; y; t)$ such that

$$\begin{aligned} v_t &= v_{xx} + \frac{\partial^2}{\partial y^2} J_\pm v; \quad x > 0; 0 < y < 1; 0 < t < 1; \\ v(0; y; t) &= J_\pm u_0(y; t); \quad 0 < y < 1; 0 < t < 1; \\ v_x(0; y; t) &= J_\pm u_1(y; t); \quad 0 < y < 1; 0 < t < 1; \end{aligned}$$

where all the \pm -mdlli⁻cations are taken with respect to $(y; t)$ and $\pm = (\pm_1; \pm_2)$.

3.1. The Scheme

Let $h > 0$ be the marching step size in the x direction. $r_{i,j}^n$, $w_{i,j}^n$, $q_{i,j}^n$, $d_{i,j}^n$ and $z_{i,j}^n$ denote the numerical approximations for $v(ih, jh; nh)$, $v_x(ih, jh; nh)$, $v_t(ih, jh; nh)$, $v_{xt}(ih, jh; nh)$ and $v_{yy}(ih, jh; nh)$ respectively. Applying the method of mdlli⁻cation, the space marching scheme to compute $r_{i,j}^n$, $w_{i,j}^n$, $q_{i,j}^n$, $d_{i,j}^n$ and $z_{i,j}^n$ is defined by

$$\begin{aligned} r_{i+1,j}^n &= r_{i,j}^n + h w_{i,j}^n; \\ w_{i+1,j}^n &= w_{i,j}^n + h (q_{i,j}^n - z_{i,j}^n); \\ q_{i+1,j}^n &= q_{i,j}^n + h d_{i,j}^n; \\ z_{i+1,j}^n &= D_y^2 (J_\pm r_{i+1}) (j \cdot h; nh); \\ d_{i+1,j}^n &= D_t (J_\pm w_{i+1}) (j \cdot h; nh); \end{aligned}$$

where n and w denote the 2D data sets $f r_{i,j}^n : 0 \leq j \leq M; 0 \leq n \leq N$ and $f w_{i,j}^n : 0 \leq j \leq M; 0 \leq n \leq N$ respectively.

The initializations for the scheme are

$$\begin{aligned} r_{0,j}^n &= J_\pm u_0^2(j \cdot h; nh); \\ w_{0,j}^n &= J_\pm u_1^2(j \cdot h; nh); \\ q_{0,j}^n &= D_t (J_\pm u_0^2) (j \cdot h; nh); \\ z_{0,j}^n &= D_y^2 (J_\pm u_0^2) (j \cdot h; nh); \\ d_{0,j}^n &= D_t (J_\pm u_1^2) (j \cdot h; nh); \end{aligned}$$

3.2. Stability of the Scheme

Without loss of generality, throughout this subsection and the next, we assume $j \cdot h_{i-1} = \min(\pm_1; \pm_2) \cdot 1$ and denote $\max_{j \in \mathbb{Z}, 0 \leq j \leq N} |y_{i,j}|$ by $jY_{i,j}$.

By Theorems 2.2 and 2.4, we have

$$\begin{aligned} j r_{i+1,j} &\leq j r_{i,j} + h j w_{i,j}; \\ j w_{i+1,j} &\leq j w_{i,j} + h (j q_{i,j} + j z_{i,j}); \\ j q_{i+1,j} &\leq j q_{i,j} + h j q_{i,j}; \\ j z_{i,j} &\leq \frac{C_1}{j \cdot h_{i-1}^2} j r_{i,j}; \\ j q_{i,j} &\leq \frac{C_2}{j \cdot h_{i-1}} j w_{i,j}; \end{aligned}$$

Therefore

$$\max_j |j r_{i+1,j}|; |j w_{i+1,j}|; |j q_{i+1,j}| \leq (1 + H) \|_+ \max_j |j r_{i,j}|; |j w_{i,j}|; |j q_{i,j}|;$$

$$\text{where } \|_+ = \max\{1 + \frac{C_1}{j \cdot h_{i-1}^2}, \frac{C_2}{j \cdot h_{i-1}}\} g.$$

Assume that the calculation stops at the i th step. Then by the iteration of the above inequality we have

$$\max_j |j r_{i,j}|; |j w_{i,j}|; |j q_{i,j}| \leq (1 + H) \|_+^i \max_j |j r_{0,j}|; |j w_{0,j}|; |j q_{0,j}|;$$

which implies

$$\max_j |j r_{i,j}|; |j w_{i,j}|; |j q_{i,j}| \leq \exp(x_i \|_+) \max_j |j r_{0,j}|; |j w_{0,j}|; |j q_{0,j}|;$$

$$\text{where } x_i = H.$$

Thus, the scheme is stable.

3.3. Convergence of the Scheme

We now consider the convergence of the scheme. In order do so, we define

$$\begin{aligned} \Phi r_{i,j}^n &= r_{i,j}^n + v(ih, jh; nh); \\ \Phi w_{i,j}^n &= w_{i,j}^n + v_x(ih, jh; nh); \end{aligned}$$

and

$$\Phi q_{i,j}^n = q_{i,j}^n + v_t(ih, jh; nh);$$

For the mdlli⁻ed solution $v(x; y; t)$, we have

$$\begin{aligned} v((i+1)h, jh; nh) &= v(ih, jh; nh) \\ &\quad + h v_x(ih, jh; nh) + O(h^2); \end{aligned}$$

$$v_x((i+1)hj; nk) = v_x(ihj; nk) + h(v_t(ihj; nk) - i \frac{\partial^2}{\partial y^2} J_{\pm} v(ihj; nk)) + O(h^2);$$

$$v_t((i+1)hj; nk) = v_t(ihj; nk) + h \frac{\partial}{\partial t} v_x(ihj; nk);$$

By comparing the equalities above with the scheme, it follows that

$$\begin{aligned} \Phi r_{i+1,j}^n &= \Phi r_{i,j}^n + h \Phi w_{i,j}^n + O(h^2); \\ \Phi w_{i+1,j}^n &= \Phi w_{i,j}^n + h \Phi q_{i,j}^n - h D_y^2(J_{\pm} r_i)(j; nk) - i \frac{\partial^2}{\partial y^2} J_{\pm} v(ihj; nk) + O(h^2); \\ \Phi q_{i+1,j}^n &= \Phi q_{i,j}^n + h D_t(J_{\pm} w_i)(j; nk) - i \frac{\partial}{\partial t} v_x(ihj; nk) + O(h^2); \end{aligned}$$

By Proposition 2.1, neglecting the effect of the $\pm j$ migration on the already modified solution v_x ,

$$\begin{aligned} &h D_t(J_{\pm} w_i)(j; nk) - i \frac{\partial}{\partial t} v_x(ihj; nk) \\ &\cdot \frac{C}{j \pm j_{i-1}} j \Phi w_{i,j} + \frac{C}{j \pm j_{i-1}} k + C_{\pm} k^2; \end{aligned}$$

Finally, using Theorems 2.3 and 2.4, we obtain

$$\begin{aligned} &h D_y^2(J_{\pm} r_i)(j; nk) - i \frac{\partial^2}{\partial y^2} J_{\pm} v(ihj; nk) \\ &\cdot \frac{C}{j \pm j_{i-1}} j \Phi r_{i,j} + \frac{C}{j \pm j_{i-1}} l + C_{\pm} l^2; \end{aligned}$$

Hence,

$$j \Phi r_{i+1,j} - j \Phi r_{i,j} + h j \Phi w_{i,j} + O(h^2);$$

$$\begin{aligned} j \Phi w_{i+1,j} - j \Phi w_{i,j} + h j \Phi q_{i,j} + \frac{C h}{j \pm j_{i-1}} j \Phi r_{i,j} + \\ \frac{C}{j \pm j_{i-1}} h + C_{\pm} h^2 + O(h^2); \end{aligned}$$

$$j \Phi q_{i+1,j} - j \Phi q_{i,j} + \frac{C h}{j \pm j_{i-1}} j \Phi w_{i,j} + \frac{C}{j \pm j_{i-1}} h k + C_{\pm} h^2 + O(h^2);$$

Setting $\Phi_i = \max\{j \Phi r_{i,j}; j \Phi w_{i,j}; j \Phi q_{i,j}\}$, we have

$$\Phi_{i+1} - (1 + \frac{C h}{j \pm j_{i-1}}) \Phi_i + \frac{C h}{j \pm j_{i-1}} (k + l) + C_{\pm} h(l^2 + k^2) + O(h^2);$$

Again, by calculating L iterations,

$$\Phi_L \leftarrow \exp\left(\frac{C X}{j \pm j_{i-1}^2}\right) (\Phi_0 + C(l + k + h));$$

Since

$$\Phi_0 \leftarrow \frac{C}{j \pm j_{i-1}^2} (l + k + h);$$

the convergence of the algorithm readily follows. That is, Φ_L converges to zero as h , k and l tend to zero.

3.4. Numerical Examples

In this section, we present numerical results obtained by applying the algorithm to two examples. The relative L_2 errors associated with the final computations are summarized in Tables 1-3. In all cases, the algorithm was implemented using the following set of parameters: $x^2 = :0.1$; $x = :3$; $p = 3$; $h = \Phi x = \Phi t = \Phi y = 1/64$; All the functions involved were approximately reconstructed in the three-dimensional domain $0 \leq x \leq :3; 0 \leq t \leq 1; 0 \leq y \leq 1$.

Example 3.1: Identify $u(x; y; t)$ in

$$\begin{aligned} u_t &= u_{xx} + u_{yy}; \\ u(0; y; t) &= \exp(t + \frac{p-1}{2}y); \\ u_x(0; y; t) &= \frac{p}{2} \exp(t + \frac{p-1}{2}y); \end{aligned}$$

The exact solution for this problem is:

$$u(x; y; t) = \exp(t + \frac{p-1}{2}(x + y));$$

Example 3.2: Identify $u(x; y; t)$ in

$$\begin{aligned} u_t &= u_{xx} + u_{yy}; \\ u(0; y; t) &= 0; \\ u_x(0; y; t) &= \exp(5|y| - 2t) \sin y; \end{aligned}$$

The exact solution for this problem is:

$$u(x; y; t) = \exp(5|y| - 2t) \sin x \sin y;$$

Table 1. GLOBAL DOMAIN ERRORS

Example	Temperature	Heat Flux
3.1	.0098	.0140
3.2	.0090	.0004

Table 2. FINALE ERRORS AT $x = :3$

Example	Temperature	Heat Flux
3.1	.0195	.1810
3.2	.0130	.0409

Table 3. BOUNDARY TEMPERATURE ERRORS

Example	Temp. at $y = 1$	Temp. at $t = 1$
3.1	.0001	.0055
3.2	.0001	.0134

ACKNOWLEDGMENT

This work was partially supported by a C. Taft Fellowship from the Taft Fund, University of Cincinnati.

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