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ESTIMATION OF HEAT SOURCES WITHIN TWO DIMENSIONAL SHAPED BODIES

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ABSTRACT

An iterative algorithm based on the conjugate gradient method is developed in order to estimate simultaneously the spatial location and the strength of heat sources within two dimensional shaped bodies. The case of punctual sources, together with temperature sensors located on the boundary of the body, is studied. The iterative regularization princilple is used to avoid the amplification of the measurement errors on the computed solution.

NOMENCLATURE

A strenght of the source
c heat capacity
$J(S)$ functional equation (8)
J_S' gradient of $J(S)$
h heat transfer coefficient
n_i normal unit vector on Γ_i
T temperature
(x_0, y_0) location of the source
Γ_i boundary surface i
λ thermal conductivity
δT sensitivity function
ψ adjoint function
ρ density

INTRODUCTION

The numerical resolution of the Direct Heat Conduction Problem (DHCP) requires the knowledge of a set of data which involves : the geometrical data (shape of the body), the medium data (heat capacity, density, thermal conductivity, heat transfer coefficient), the " source " data (heat flux density, fixed temperature on the boundary, heat sources within the body) , the initial condition (spatial distribution of the temperature field at the initial time). When some part of this set of data is unknown, the DHCP solution cannot be computed, the numerical resolution of the Inverse Heat Conduction Problems (IHCP) aims to determine the unknown part of these data from additional information. These new data are usually given by temperature sensors which are located on the boundary or inside the body. During the last decade, due to the tremendous advancement in scientific computation, an increasing attention has been devoted to the techniques for solving multidimensional IHCP. Numerical resolutions of IHCP are well known to be ill - conditioned, therefore some regularizing process has to be considered to avoid numerical instabilities on the computed solutions. Several approaches are available (Alifanov, 1994), (Jarny et al, 1991), (Beck, 1977). A general approach consists in solving IHCP with numerical optimization tools like the conjugate gradient method. Combined with the use of a Finite Element Library, It is a powerful method which offers a wide field of practical applications in inverse thermal analysis. In this work, some results are presented on the estimation of heat sources within two dimensional shaped bodies. This problem has been considered by several authors (Silva Neto et al, 1992), (Le Niliot, 1998),(Park H.M. et al, 1999) especially in the case of punctual sources and assuming the spatial location is known. In this paper it is shown how the simultaneous determination of the spatial location and the strength of ponctual heat sources can be achieved from temperature histories delivered by sensors located on the boundary of the body. Numerical experiments show that the influence of the error measurements

on the computed solution can be strongly reduced according to the iterative regularization principle.

DIRECT HEAT CONDUCTION PROBLEM (DHCP)

The heat conduction process involving heat source is considered within an arbitrary shaped domain Ω with the boundary $\Gamma_1 \cup \Gamma_2$. The set of equations of the DHCP are written in the form of a parabolic problem

$$\rho c \frac{\partial T}{\partial t} - \lambda \Delta T = S(x,y,t) \text{ in } \Omega,\ t > 0 \tag{1}$$

$$\lambda \frac{\partial T}{\partial n_1} + hT = hT_e \text{ on } \Gamma_1,\ t > 0 \tag{2}$$

$$\lambda \frac{\partial T}{\partial n_2} = 0 \text{ on } \Gamma_2,\ t > 0 \tag{3}$$

$$T = T_0 \text{ in } \Omega,\ t = 0 \tag{4}$$

Solving the DHCP consists in the determination of the temperature field $T(x,y,t)$ on the domain Ω, at each time t, when the medium data ρC, λ , the heat transfer coefficient h, the external temperature T_e, the heat source $S(x,y,t)$ and the initial field T_0, are assumed to be known. In this study, the heat source is considered with the following form

$$\begin{gathered} S(x,y,t) = A(t).F(x,y;x_0,y_0) \\ \int_\Omega F d\Omega = 1 \end{gathered} \tag{5}$$

where A and F are respectively the strength and the spatial distribution of the source.

The goal is to determine the strength $A(t)$, and the location $(x_0,\ y_0)$ for a prescribed spatial distribution F.

PONCTUAL HEAT SOURCE

The spatial distribution of the heat source to be determined is described by a gaussian function

$$S(x,y,t) = \frac{P(t)}{\pi\omega^2} \exp\Big(- \frac{(x-x_0)^2 - (y-y_0)^2}{\omega^2} \Big) \tag{6}$$

where $P(t)$ is the intensity (W/m) and ω is a parameter used to simulate a punctual heat source, it is chosen equal to the mesh size. We consider

$$A(t) = \frac{P(t)}{\pi\omega^2} \tag{7}$$

and $(x_0,\ y_0)$ is the source location.

FORMULATION OF THE IHCP

Additional information are provided by Ns sensors located within the spatial domain or on its boundary, on the time interval $[0,\ t_f]$. Let $Y_m(t)$, $(m = 1,..Ns)$ be the temperature histories delivered at the location $(x_m,\ y_m)$. Solving the IHCP consists in the determination of the source S which matches the solution of the DHCP $T(x_m,y_m,t;S)$ with the data $Y_m(t)$. Because of ill-posedness, the IHCP is solved in the least square sense, by minimizing the residual functional

$$J(S) = \frac{1}{2} \int_0^{t_f} \sum_{m=1}^{Ns} |T(x_m,y_m,t;S) - Y_m(t)|^2 dt \tag{8}$$

with $T(x,y,t;S)$ solution of equations (1)-(4)

MINIMIZATION OF THE FUNCTIONAL $J(S)$

Minimization of $J(S)$ is achieved by using the conjugate gradient method (CGM). Let S be the unknown vector to be determined.

$$S^T = [x_0, y_0, A_{k\,(k=0,..Nt)}] \tag{9}$$

where Nt is the number of time step. The (CGM) is iterative, at each iteration, the previous estimate S^n is corrected by

$$S^{n+1} = S^n - \gamma_n d^n \tag{10}$$

where the search direction d^n and the step length γ_n are determined in order to have

$$J(S^{n+1}) < J(S^n) \tag{11}$$

The vector $[d_{x_0}, d_{y_0}, d_{A_k\,(k=0,..Nt)}]$, is chosen according to the CGM rules

$$\begin{aligned} d^n &= J_S'^{\,n} + \beta^n J_S'^{\,n-1} \\ \beta^0 &= 0 \\ \beta^n &= \frac{< J_S'^{\,n}, J_S'^{\,n} - J_S'^{\,n-1} >}{\|J_S'^{\,n}\|^2} \end{aligned} \tag{12}$$

where $J_S^{'n}$ is the gradient of the $J(S)$ at the iteration n

$$J_S^{'} = [J_{x_0}^{'}, J_{y_0}^{'}, J_{A_k\,(k=0,\ldots Nt)}^{'}] \tag{13}$$

$||.||$ is the vector norm associated to the scalar product $< .,. >$ The step length γ_n is given by

$$\gamma_n = \frac{\sum_{m=1}^{Ns} \int_0^{t_f} \delta T(x_m, y_m, t; S^n)[T(x_m, y_m, t; S^n) - Y_m(t)]dt}{[\delta T(x_m, y_m, t; S^n)]^2} \tag{14}$$

where δT is the sensitivity function which results of the variation δS of the source.

SENSITIVITY EQUATIONS

Solving the sensitivity equations consists in the determination of the temperature variation δT which results of a variation δS of the source. The linearity of the equations (1)-(4) leads to

$$\begin{gathered} \rho c \frac{\partial \delta T}{\partial t} - \lambda \Delta \delta T = \delta S \\ \delta S = F \delta A + A\left(\frac{\partial F}{\partial x_0}\delta x_0 + \frac{\partial F}{\partial y_0}\delta y_0\right) \text{ in } \Omega,\ t > 0 \end{gathered} \tag{15}$$

$$\lambda \frac{\partial \delta T}{\partial n_1} + h\delta T = 0 \text{ on } \Gamma_1,\ t > 0 \tag{16}$$

$$\lambda \frac{\partial \delta T}{\partial n_2} = 0 \text{ on } \Gamma_2,\ t > 0 \tag{17}$$

$$\delta T = 0 \text{ in } \Omega,\ t = 0 \tag{18}$$

and the associated variation of the functional $J(S)$ is

$$\delta J(S) = \sum_m^{Ns} \int_0^{t_f} \int_\Omega (T(x,y,t;S) - Y_m(t))\delta(x - x_m)\delta(y - y_m)\delta T dt d\Omega \tag{19}$$

where $\delta(.)$ is the Dirac impulse. By definition, $J_S^{'}(x,y,t;S)$ satisfies

$$\delta J(S) = \int_0^{t_f} \int_\Omega J_S^{'}(x,y,t;S)\delta S dt d\Omega \tag{20}$$

In order to get $J_S^{'}(x,y,t;S)$, equation (19) has to be put in the form of equation (20). This is done by making stationary the Lagrangian associated to the optimization problem, equation (8)

LAGRANGIAN AND ADJOINT EQUATIONS

Let us introduce the Lagrangian $L(T,S,\psi)$

$$\begin{gathered} L((T,S,\psi)) = \\ \tfrac{1}{2}\int_0^{t_f} \int_\Omega \sum_{m=1}^{Ns} (T(x,y,t;S) - Y_m(t))^2 \delta(x - x_m)\delta(y - y_m) dt d\Omega \\ + \int_0^{t_f} \int_\Omega (\rho c \frac{\partial T}{\partial t} - \lambda \Delta T - S)\psi dt d\Omega \end{gathered} \tag{21}$$

where T, S, ψ can be independent functions. The Lagrangian multiplier ψ is, as the function T, a function of x, y, and t. ψ being fixed, the differential of L satisfies

$$\delta L = \frac{\partial L}{\partial T}\delta T + \frac{\partial L}{\partial S}\delta S \tag{22}$$

Let us take ψ solution of the following equation (so called adjoint equation)

$$\frac{\partial L}{\partial T}\delta T = 0, \qquad \forall\ \delta T \tag{23}$$

then by taking T solution of the equations (1)-(4), it comes

$$\delta J = \delta L = \frac{\partial L}{\partial S}\delta S \tag{24}$$

where

$$\begin{aligned} \frac{\partial L}{\partial S}\delta S &= -\int_0^{t_f} \int_\Omega \psi \delta S\, dt d\Omega \\ &= -\int_0^{t_f} \int_\Omega \psi F \delta A\, dt d\Omega - \int_0^{t_f} \int_\Omega \psi A \frac{\partial F}{\partial x_0}\delta x_0\, dt d\Omega \\ &\quad - \int_0^{t_f} \int_\Omega \psi A \frac{\partial F}{\partial y_0}\delta y_0\, dt d\Omega \end{aligned} \tag{25}$$

and the gradient components $J_S^{'}(x,y,t;S)$ are easily extracted from equation (25),

$$J_A^{'}(t) = -\int_\Omega \psi F\ d\Omega \tag{26}$$

$$J_{x_0}^{'} = -\int_0^{t_f} \int_\Omega \psi A \frac{\partial F}{\partial x_0}\, dt d\Omega \tag{27}$$

$$J'_{y_0} = -\int_0^{t_f}\int_\Omega \psi A \frac{\partial F}{\partial y_0}\, dt d\Omega \tag{28}$$

It can be noted that $J_A{}'$ is a function of time, like the function A(t), $J_{x_0}{}'$ and $J_{y_0}{}'$ are real numbers like x_0 and y_0.

Let us develop equation (23). From the definition of the Lagrangian, (21), we have

$$\begin{aligned}\frac{\partial L}{\partial T}\delta T = \\ \int_0^{t_f}\int_\Omega \sum_{m=1}^{Ns}(T(x,y,t;S)-Y_m(t))\delta(x-x_m)\delta(y-y_m)\delta T dt d\Omega \\ + \int_0^{t_f}\int_\Omega (\rho c\frac{\partial \delta T}{\partial t} - \lambda\Delta\delta T)\psi dt d\Omega\end{aligned} \tag{29}$$

integration by parts give

$$\int_0^{t_f}\frac{\partial \delta T}{\partial t}\psi dt = [\psi\delta T]_{t=0}^{t=t_f} - \int_0^{t_f}\frac{\partial \psi}{\partial t}\delta T dt \tag{30}$$

$$\begin{aligned}\int_\Omega \lambda\Delta\delta T\ \psi d\Omega = \int_{\Gamma_1}(\lambda\frac{\partial\psi}{\partial n_1} + h\psi)\delta T d\Gamma_1 + \int_{\Gamma_2}\lambda\frac{\partial\psi}{\partial n_2}\delta T d\Gamma_2 \\ + \int_\Omega \lambda\Delta\psi\ \delta T d\Omega\end{aligned} \tag{31}$$

then with the boundary and initial conditions (16), (17), (18), equation (29) can be written in the new form

$$\begin{aligned}\frac{\partial L}{\partial T}\delta T = \int_\Omega\int_0^{t_f}(-\rho c\frac{\partial\psi}{\partial t} - \lambda\Delta\psi)\delta T\ dt d\Omega \\ + \int_\Omega\int_0^{t_f}\sum_{m=1}^{Ns}(T(x,y,t;S)-Y_m(t))\delta(x-x_m)\delta(y-y_m)\delta T\ dt d\Omega \\ + \int_{\Gamma_1}\int_0^{t_f}(\lambda\frac{\partial\psi}{\partial n_1} + h\psi)\delta T\ dt d\Gamma_1 \\ + \int_{\Gamma_2}\int_0^{t_f}\lambda\frac{\partial\psi}{\partial n_2}\delta T\ dt d\Gamma_2 + \int_\Omega \psi(t=t_f)\delta T\ d\Omega\end{aligned} \tag{32}$$

Therefore equation (23) is satisfied by taking the lagrange multiplier ψ solution of the following set of equations

$$\begin{aligned}-\rho c\frac{\partial\psi}{\partial t} - \lambda\Delta\psi = \\ \sum_{m=1}^{Ns}(T(x,y,t;S)-Y_m(t))\delta(x-x_m)\delta(y-y_m)\ \text{ in } \Omega,\ t>0\end{aligned} \tag{33}$$

$$\lambda\frac{\partial\psi}{\partial n_1} + h\psi = 0 \text{ on } \Gamma_1,\ t>0 \tag{34}$$

$$\lambda\frac{\partial\psi}{\partial n_2} = 0 \text{ on } \Gamma_2,\ t>0 \tag{35}$$

$$\psi = 0 \ \text{ in } \Omega,\ t=t_f \tag{36}$$

Equations (33)-(36) are linear, the solution of which ψ is computed backward in time on the interval $[0,t_f]$

NUMERICAL ALGORITHM

Each iteration of the CGM includes the numerical resolution of three distinct parabolic set of equations. The iterative algorithm has the following structure

a) Choose an initial guess S^0
b) Solve the direct problem, equations (1)-(4), to compute $T(x_m,y_m,t;S^n)$, and $J(S^n)$, equation (8)
c) Solve the adjoint problem ,equations (33-36) to compute the vector J'_{S^n}, equations (26-28)
d) Solve the sensitivity problem, equations (15-18), to compute γ_n, equation (14), and the new vector S^{n+1} equation (10)
e) Stopping condition
if $\left(J(S^{n+1}) > J(S^n) \text{ or } J(S^n) < \varepsilon\right)$ go to f)
else set $n = n+1$ go to b)
f) End

ε is defined according to the iterative regularization principle which is illustrated in the following section.

NUMERICAL EXPERIMENTS

The spatial domain considered is a square, see figure (1), $L = 0.05m$. The values of the thermophysical parameters are : $\lambda = 0.17W/K.m$, $a = 1.210^{-07}m^2/s$, $h = 10W/m^2.K$, the initial temperature $T_0 = 293K$, and the external temperature $T_e = 293K$. A regular mesh with $11\times 11 = 121$ nodes is considered. $\omega = 6.10^{-3}m$. The parabolic solver of the Finite Element Library MODULEF is used to compute the solution of the DHCP.

Two cases have been studied :
case#1: $x_0 = 0.01m$, $y_0 = 0.04m$
case#2: $x_0 = 0.03m$, $y_0 = 0.01m$

The temperature responses delivered by four sensors located at the middle of each side of the domain,to a time varying strength A(t) of the source, are shown on figures (2) and (3).

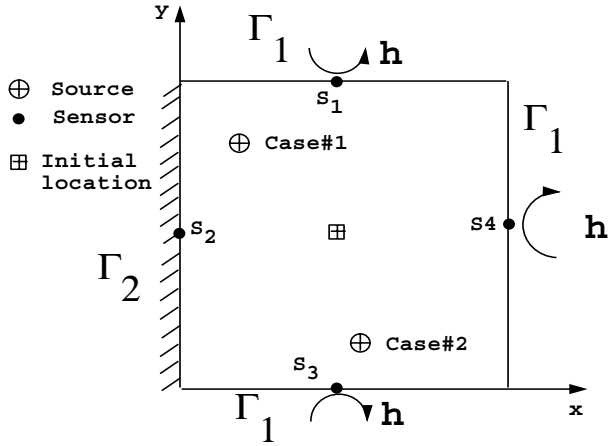

Figure 1. DOMAIN Ω

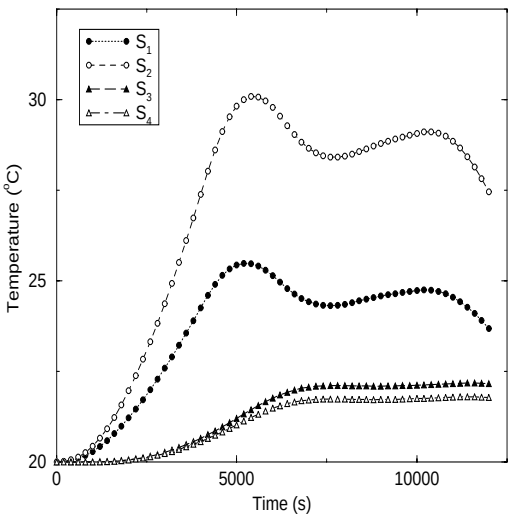

Figure 2. TEMPERATURE RESPONSES : case#1

These responses are corrupted by adding a zero mean gaussian noise, to build the simulated additional data $Y_m(t)$ required to solve the IHCP.

$$Y_m^k = T_m^k exact + \sigma e_m^k \qquad m = 1, \ldots Ns, \; k = 0, \ldots Nt \qquad (37)$$

where σ is the standard deviation of the noise (assumed to be identical for each sensor) and e_m^k is the normally distributed randum number.

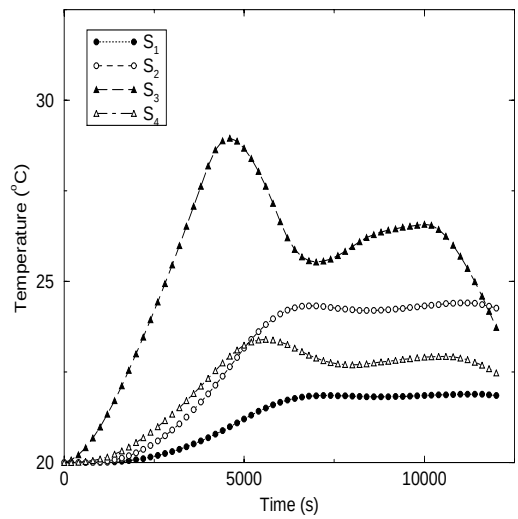

Figure 3. TEMPERATURE RESPONSES : case#2

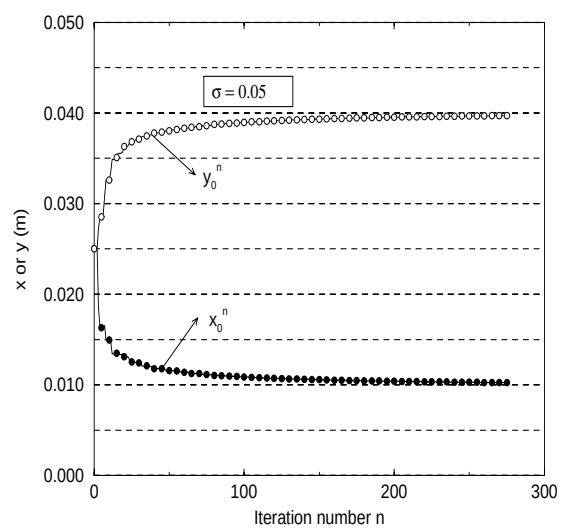

Figure 4. IDENTIFICATION OF THE LOCATION : case#1, $\sigma = 0.05$

NUMERICAL RESULTS

- Case#1

The initial guess of the source location is taken at the center of the square $x = 0.025m$ $y = 0.025m$. The initial guess of the strength of the source is taken equal to zero. An alternate iterative research process of the conjugate gradient has been performed : instead of modifying the location and the strength at

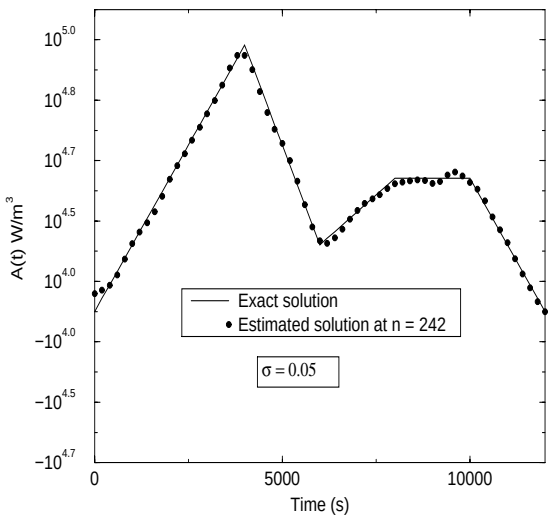

Figure 5. IDENTIFICATION OF THE STRENGTH : case#1, $\sigma = 0.05$

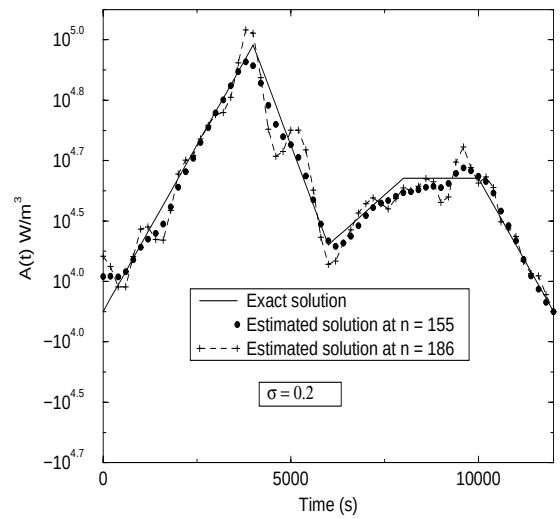

Figure 7. IDENTIFICATION OF THE STRENGTH : case#1, $\sigma = 0.2$

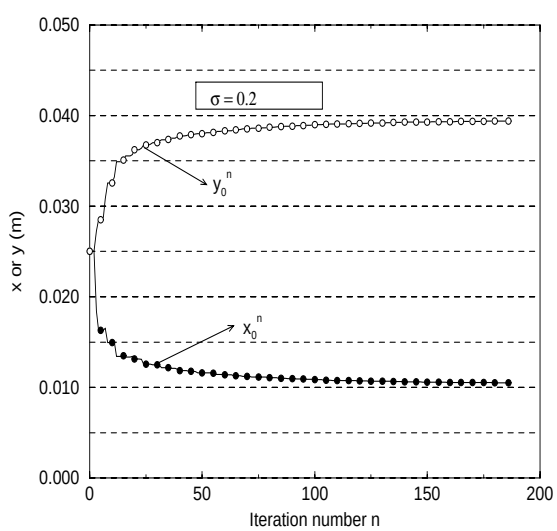

Figure 6. IDENTIFICATION OF THE LOCATION : case#1, $\sigma = 0.2$

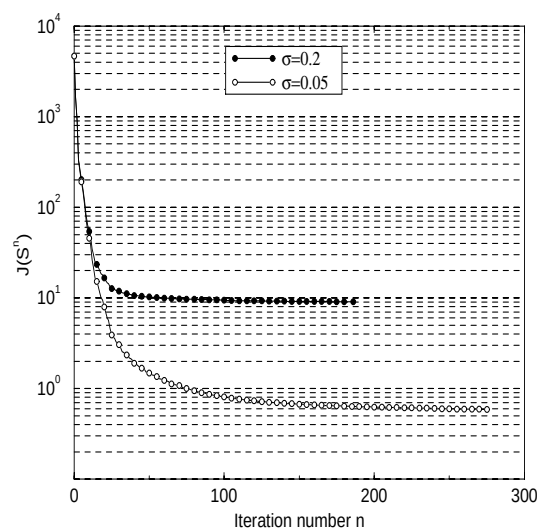

Figure 8. MINIMIZATION OF $J(S^n)$: case#1

each iteration, one iteration is used to correct the location keeping the strength unmodified, the following iteration is used to correct the strength while the location is hold, and so on. On figure (4) and (6) are shown the computed locations resulting of this minimization process, with a noise level $\sigma = 0.05$ and $\sigma = 0.2$ respectively. Figure (8) illustrates the convergence of the sequence $J(S^n)$: asymptotic values J_{as} are observed, they are directly related to the noise level σ as indicate in Table 1. The final number of iterations n^* is chosen according to the following equation

$$J_{as} = J(S^{n^*}) = Nt.Ns.\sigma^2 = \varepsilon \tag{38}$$

When the iterative process is stopped at the iteration number n^*,

Table 1. RESULTS OF IDENTIFICATION OF (x_0, y_0), case#1

σ	ε	J_{as}	n^*	x_0	y_0
0.05	0.6	0.5990	242	0.0102	0.0396
0.2	9.6	9.6049	155	0.0110	0.0387

Table 2. RESULTS OF IDENTIFICATION OF (x_0, y_0), case#2

σ	ε	J_{as}	n^*	x_0	y_0
0.05	0.6	0.6235	63	0.0299	0.0099
0.2	9.6	9.5908	59	0.0296	0.0098

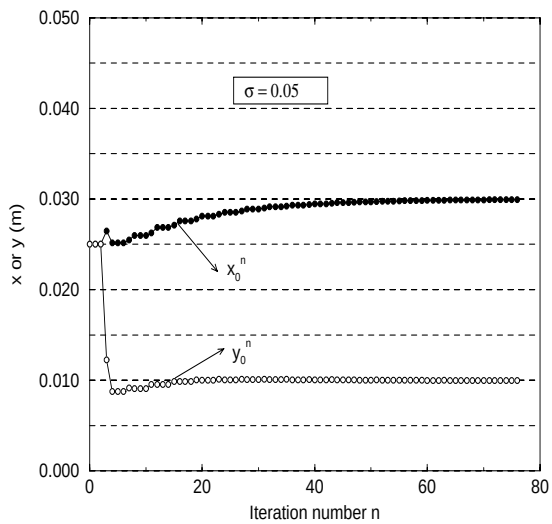

Figure 9. IDENTIFICATION OF THE LOCATION : case#2, $\sigma = 0.05$

no instability occur on the computed solution as it is shown on figure (5). Figure (7) illustrates the iterative regularization principle :

- with $\sigma = 0.2$ and $n^* = 155$, the computed strength $A(t)$ is stable, close to the exact values; if the iterative process is continued, for example $n = 186$, oscillations appear.
- with a lower noise level $\sigma = 0.05$, the final number $n^* = 242$ is greater and the computed strength is more accurate figure (5)

• Case#2

The same initial guess is considered. Figures (9) and (11) illustrates the convergence of the computed source locations to the exact values. Figures (10) and (12) show the influence of the noise level on the computed strength $A(t)$. Results are similar to

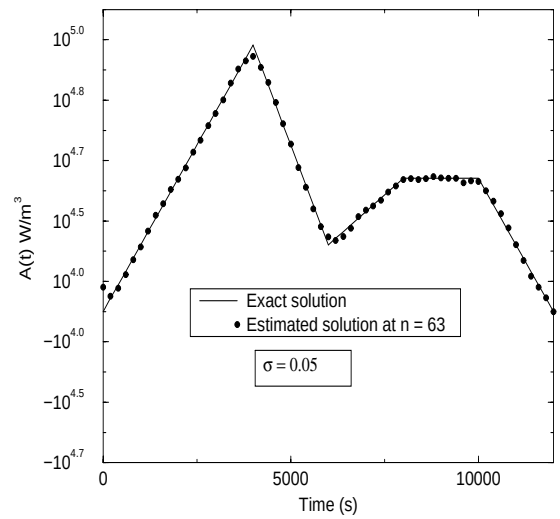

Figure 10. IDENTIFICATION OF THE STRENGTH : case#2, $\sigma = 0.05$

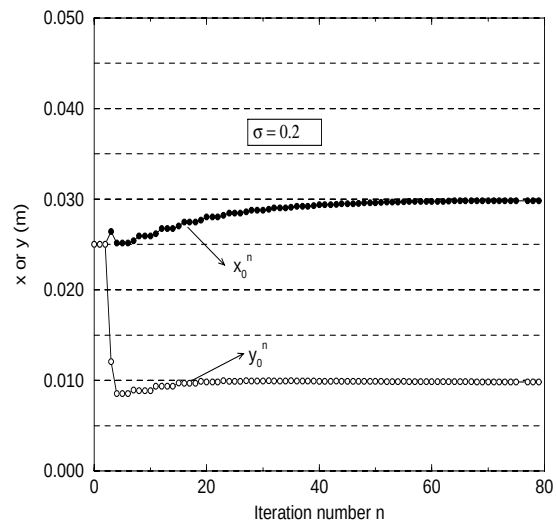

Figure 11. IDENTIFICATION OF THE LOCATION : case#2, $\sigma = 0.2$

case#1. It can be observed that the final numbers of iterations depends on the level noise, like in the previous case. The influence of the sensor locations must be also noted. The distance between the nearest sensor and the source is lower than in case#1, then for $\sigma = 0.05$, instead of $n^* = 242$ only $n^* = 63$ iterations are required, and for σ=0.2, instead of $n^* = 155$, $n^* = 59$ are sufficient table 2.

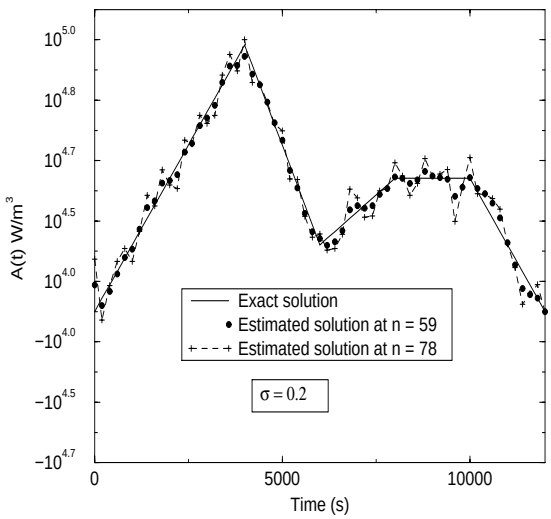

Figure 12. IDENTIFICATION OF THE STRENGTH : case#2, $\sigma = 0.2$

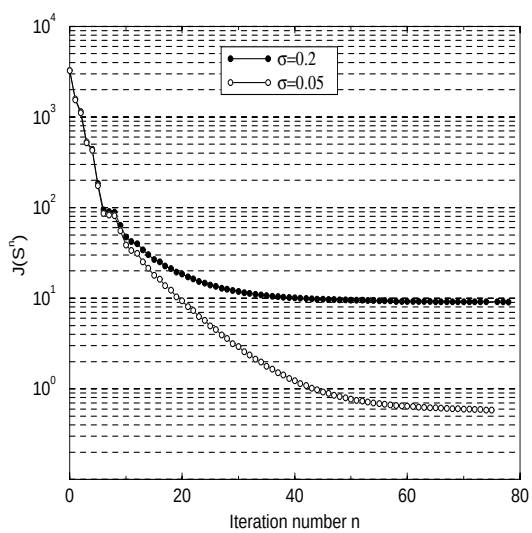

Figure 13. MINIMIZATION OF $J(S^n)$: case#2

CONCLUSION

The inverse heat conduction problem which consists in the determination of the location and simultaneously the time varying strength of a heat ponctual source within a two dimensional arbitrary shaped body, has been computed by the conjugate gradient method, using a parabolic solver of the Finite Element Library MODULEF. The spatial distribution of the source has been described by a stiff gaussian function in order to approximate a punctual distribution. The iterative regularization principle together with the use of the conjugate gradient method has been illustrated. To avoid instabilities on the computed strength, the iterative process of minimization has to be stopped when the residual functional reaches a critical value which depends on the noise level.

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