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NEW RESULTS ON THE NON-LINEARITY AND THE SENSITIVITY OF THE ESTIMATION OF THE DIFFUSION COEFFICIENT IN A 2D ELLIPTIC EQUATION

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ABSTRACT

We consider the problem of determining the diffusion coefficient a(x) in a 2D elliptic equation from a distributed measurement z in H^1 of the solution u of the equation. For a problem with a simple geometry, we give conditions under which the first derivative of the $b = 1/a \mapsto u$ mapping is coercive. Then we show that its non linearity in a direction d increases, and its sensitivity decreases, when the ratio $|\nabla(d/b)|_{L^2}/|d/b|_{L^2}$ increases. This corroborates observations on scale, sensitivity and non linearity made in (Chavent and Liu, 89) (Grimstad and Mannseth, 99).

1 INTRODUCTION

The estimation of the distributed diffusion coefficient a in the elliptic equation

$$-\nabla(a\nabla u) = F \quad \text{in }\Omega\tag{1}$$

is a long studied problem. In equation (1), $\Omega \subset \mathbb{R}^2$ is the space domain, and *F* is a supposedly known source function.

The $a \mapsto u$ mapping is very non-linear, and its inversion from measurement data z on u by a least-squares fitting technique often unstable when data are noisy. Stabilization can be obtained by regularization, or by using a multiscale approach, which also enhances the performance of the optimizer (Chavent and Liu,

89) (Liu, 93). It was shown in (Liu, 93) for a one-dimensional problem with constant coefficcient that the non-linearity of the $a \mapsto u$ mapping was increasing in the direction of finer details of the parameter space, when at the same time the sensitivity was decreasing. This relation between nonlinearity, scale and sensitivity was investigated further recently in (Grimstad and Mannseth, 99), also for a 1D problem. We extend in this paper the previous results to a two-dimensional case with a simple geometry. Our main result is that, under technical assumption which will be made clear below, the curvature of the $b = 1/a \mapsto u \in H^1(\Omega)$ mapping in a direction d of the parameter space is bounded from above by an increasing function of $|\nabla(d/b)|_{L^2(\Omega)}/|d/b|_{L^2(\Omega)}$, which vanishes at zero. Simultaneously, the sensitivity in the direction d is bounded from below by a decreasing function of $|\nabla(d/b)|_{L^2(\Omega)}/|d/b|_{L^2(\Omega)}$, which goes to zero when the upper bound on the curvature blows up to infinity. This implies immediately, when a multiscale parametrization of b is used, that this curvature increases when the parametrization of b is changed from coarse to fine.

2 SETTING OF THE PROBLEM

We want to recover *a* in (1) as a function of $x \in \Omega$ from some measurement *z* of the solution *u*. In most practical application, the available measurement *z* are very lacunary (point or boundary data), but one would already be content to handle the case where *z* is a measurement of *u* in $L^2(\Omega)$. We shall nevertheless consider

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in this paper the application wise unlikely case where

z is a measurement of u in the "state space" $H^1(\Omega)$ of the elliptic equation.

The properties of the $a \mapsto u \in H^1(\Omega)$ mapping we obtain in this paper will then allow to use the state-space regularization approach of (Chavent and Kunisch, 93), (Chavent and Kunisch, 98) to handle the more realistic case of an L^2 observation of u. We shall consider the simple case where the source term F of (1) is made of a distributed source $f \in L^2(\Omega)$ and single Dirac-like sources with given flow rate. It will be convenient to implement theses sources as boundary conditions over the boundaries γ of small holes in Ω . We shall denote by G the collection of the boundaries γ of each hole, and by $Q_{\gamma} \in \mathbb{R}$ the given flow rate assigned to the source $\gamma \in G$. Hence we replace (1) by

$$-\nabla (a\nabla u) = f \quad \forall x \in \Omega, \tag{2}$$

$$\int_{\gamma} a \frac{\partial u}{\partial v} = Q_{\gamma}, u|_{\Gamma} = u_{\gamma} \in \mathbb{R} \text{ (unknown constant), } \forall \gamma \in G. \quad (3)$$

which we complement by Dirichlet conditions on the outer boundary Γ of $\partial \Omega$:

$$u = 0 \quad \text{on } \Gamma. \tag{4}$$

Remark 2.1. All what follows can be easily adapted to handle the case of Neumann conditions on Γ :

$$a\frac{\partial u}{\partial v} = g \text{ on } \Gamma \tag{5}$$

provided the source terms Q_{γ} , f and g satisfy:

$$\sum_{\gamma \in G} \mathcal{Q}_{\gamma} + \int_{\Omega} f + \int_{\Gamma} g = 0 \tag{6}$$

and (2)(3)(5) is complemented by the additional equation

$$\int_{\Omega} u = 0. \tag{7}$$

We consider now the set of admissible parameters. Rather than searching for the diffusion coefficient *a* itself, we shall search for

its reciprocal b = 1/a. This reduces somewhat the non-linearity of the parameter \rightarrow output mapping (think of the case $a = \text{con$ $stant !}$). It allows to obtain stability estimates in the 1D case (Chavent and Kunisch, 93), and proves necessary also in the 2D case we consider here. Another similarity with the 1D case, where *b* had to be constant over a small interval surrounding the sources, we shall require in 2D that *b* is constant on each source boundary γ . This is a physically reasonable assumption as the size of the boundary γ is small compared to the scale at which we hope to recover b = 1/a. So our choice for the set *D* of admissible parameters is:

$$D = \{ b \in H^1(\Omega) | | \\ 0 < b_m \le b(x) \le b_M a.e. \text{ on } \Omega, b|_{\gamma} = b_{\gamma} \in \mathbb{R} \}$$

$$(8)$$

where b_m, b_M are given positive constants.

Then for any $b \in D$, the variational formulation of (2) (3)(4) is :

$$u \in V \text{ s.t. } \int_{\Omega} \frac{1}{b} \nabla u \cdot \nabla v = \int_{\Omega} f v + \sum_{\gamma \in G} Q_{\gamma} v_{\gamma} \quad \forall v \in V, \quad (9)$$

where

$$V = \{ v \in H^{1}(\Omega) | v|_{\gamma} = v_{\gamma} = \text{ unknown constant } \forall \gamma \in G \}.$$
(10)

Equation (9) defines the non-linear mapping $b \in D \mapsto u(b) \in H^1(\Omega)$ to be inverted. The corresponding least squares problem is:

$$\hat{b}$$
 minimizes $J(b) = |\nabla(u(b) - z)|^2_{L^2(\Omega)}$ on D (11)

where

$$z \in H^1(\Omega)$$
 is a given measurement of u . (12)

In order to estimate the curvature of the $b \mapsto u(b)$ mapping in a direction $d = b_1 - b_0$ ($b_0, b, \in D$), we shall require that the solution *u* of (9) satisfies the following inequality:

$$|\nabla u(x)| < M \ a.e. \ \text{on } \Omega, \forall b \in D \tag{13}$$

where *M* is independent of *b*.

This will clearly be satisfied if for example f = 0. Finally, we shall also suppose that

$$\Omega$$
 connected, $\partial \Omega = \Gamma \cup (U_{\gamma \in G})$ is regular (14)

which implies, by a regularity theorem, that *u* is in $H^2(\Omega)$, and hence in $C(\overline{\Omega})$ as n = 2. This implies that

$$|u(x)| \le u_M \ a.e. \text{ on } \Omega, \forall b \in D \tag{15}$$

where u_M is independent of *b*. Also, if we define

$$W = \{ v \in H^1(\Omega) | \int_{\Omega} v = 0 \},$$

then for any $g \in L^2(\Omega)$ such that $\int_{\Omega} g = 0$, the elliptic equation

$$\int_{\Omega} b\nabla w \nabla v = \int_{\Omega} gv \quad \forall v \in W$$
(16)

has a unique solution $w \in W$. As $\partial \Omega$ is regular by (14) and $b \in H^1(\Omega)$ by (8), we see that the solution w of (16) is in fact in $H^2(\Omega)$, and hence in $L^{\infty}(\Omega)$. We shall denote by c_{∞} the corresponding continuity constant:

$$b_m \|w\|_{\infty} \le c_{\infty} |g|_{L^2(\Omega)},\tag{17}$$

where c_{∞} is independent of *b*.

3 A VECTOR FIELD DECOMPOSITION OF $L^2(\Omega) \times L^2(\Omega)$

A basic ingredient to our proof will be a div-rot decomposition of $\mathbb{L}^2(\Omega) = L^2(\Omega) \times L^2(\Omega)$ which is adapted to the elliptic equation (2) (3) (4) or (9).

We associate to V an equivalence relation \simeq on vector fields \vec{q} of $\mathbb{L}^2(\Omega)$ by:

$$\vec{q} = \vec{q}' \iff \int_{\Omega} \vec{q} \cdot \nabla v = \int_{\Omega} \vec{q}' \nabla v \quad \forall v \in V$$
(18)

and we denote by

$$\begin{cases} G = \mathbb{L}^2(\Omega) \simeq \text{ the corresponding quotient space} \\ G^{\perp} \text{ its orthogonal complement in } \mathbb{L}^2(\Omega) \end{cases}$$
(19)

Then one can prove, by techniques similar to (Girault and Raviart, 86):

Proposition 3.1. *The space* G *and* G^{\perp} *are given by:*

$$G = \{\nabla \varphi \quad , \varphi \in V\}$$
(20)

$$G^{\perp} = \{ \nabla \land \Psi \quad , \Psi \in W \}$$
(21)

where W is defined by (16) and where

$$\nabla \phi = gradient \, of \phi$$
,

 $\nabla \wedge \Psi = rotational of \Psi.$

Let now

$$b \in D$$
 , $d \in H^1(\Omega)$ (22)

be a nominal parameter and a perturbation direction in the admissible set D, and

$$u = u(b) \in V \tag{23}$$

be the corresponding solution of (9), which is supposed to satisfy (13) and (15). By virtue of (20), we see of course that

$$\nabla u \in G. \tag{24}$$

If the function *d* is constant, then $d\nabla u \in G$ too ! But in general $d\nabla u \notin G$; it has a non-zero component $\nabla \wedge \Psi$ on G^{\perp} , given by:

$$\Psi \in W, \int_{\Omega} \nabla \wedge \Psi. \nabla \wedge v = \int_{\Omega} d\nabla u. \nabla \wedge v \quad \forall v \in W.$$

One can prove then the

Proposition 3.2. (Hypothesis and notation (22) (23)):

$$\frac{|d\nabla u|_{G^{\perp}}}{|d\nabla u|_{L^2}} = \frac{|\nabla \wedge \Psi|_{L^2}}{|d\nabla u|_{L^2}} \le C_W(\Omega) \frac{|\nabla \wedge d \cdot \nabla u|_{L^2}}{|d\nabla u|_{L^2}}$$
(25)

where:

$$C_W(\Omega) = Poincaré constant for the space W.$$
 (26)

Proposition 3.2 shows that the relative amplitude of the component of $d\nabla u$ on G^{\perp} tend to be small when *d* is "not too far from a constant", this "distance" being measured by the relative amplitude of $|\nabla \wedge d.\nabla u|_{L^2}$ to $|d\nabla u|_{L^2}$.

4 AN INVERSE STABILITY ESTIMATE

Let $b_0, b_1 \in D$ be two admissible parameters, and $u_0, u_1 \in V$ the corresponding solutions of (9). One obtains by substraction:

$$\int_{\Omega} \frac{b_1 - b_0}{b_0 b_1} \nabla u_0 \cdot \nabla v = \int_{\Omega} \frac{1}{b_1} (\nabla u_0 - \nabla u_1) \nabla v \quad \forall v \in V, \quad (27)$$

for which the following inverse stability estimate holds:

Proposition 4.1. If

$$d = \frac{b_1 - b_0}{b_0 b_1} \tag{28}$$

satisfies

$$C_W(\Omega) \frac{|\nabla \wedge d. \nabla u_0|_{L^2}}{|d\nabla u_0|_{L^2}} < 1,$$
(29)

where $C_W(\Omega)$ is defined in (26), then:

$$\left(1 - C_W(\Omega)^2 \frac{|\nabla \wedge d.\nabla u_0|_{L^2}^2}{|d\nabla u_0|_{L^2}^2}\right)^{\frac{1}{2}} |d\nabla u_0|_{L^2} \le |\frac{1}{b_1} (\nabla u_0 - \nabla u_1)|_{L^2}.$$
(30)

Proof. Equation (27) rewrites, using the equivalence relation \simeq defined above:

$$d\nabla u_0 \simeq \frac{1}{b_1} (\nabla u_0 - \nabla u_1),$$

so that the two norms in the quotient space G are equal:

$$|d\nabla u_0|_G = |\frac{1}{b_1}(\nabla u_0 - \nabla u_1)|_G.$$
 (31)

The orthogonal decomposition of $d\nabla u_0$ on $G \oplus G^{\perp}$ is:

$$d\nabla u_0 = \nabla \phi + \nabla \wedge \Psi \quad , \phi \in V, \Psi \in W,$$

so that

$$|d\nabla u_0|^2_{L^2} = |\nabla \varphi|^2_{L^2} + |\nabla \wedge \Psi|^2_{L^2}$$

Hence:

$$|d\nabla u_0|_G^2 = |\nabla \varphi|_{L^2}^2 = |d\nabla u_0|_{L^2}^2 - |\nabla \wedge \Psi|_{L^2}^2.$$

But *d* defined by (28) satisfies $d \in H^1(\Omega)$, and we can apply proposition 3.2. Hence

$$|d\nabla u_0|_G^2 \ge |d\nabla u_0|_{L^2}^2 \left(1 - C_W(\Omega)^2 \frac{|\nabla \wedge d.\nabla u_0|_{L^2}^2}{|d\nabla u_0|_{L^2}^2}\right).$$
 (32)

By definition of the quotient space G one has also:

$$|\frac{1}{b_1}(\nabla u_0 - \nabla u_1)|_G \le |\frac{1}{b_1}(\nabla u_0 - \nabla u_1)|_{L^2}.$$
 (33)

Combining (31) (32) (33) under hypothesis (29) gives the estimation (30).

5 FINITE CURVATURE ESTIMATES

We analyse in this section the velocity η and the acceleration ξ in the data space $H^1(\Omega)$ along the path *P* image by the $b \mapsto u(b)$ mapping of a given $[b_0, b_1]$ segment of *D*. Given $b_0, b_1 \in D$ and $t \in [0, 1]$ we set:

$$b_t = (1-t)b_0 + tb_1 \in D, \quad u_t = u(b_t) \in H^1(\Omega).$$
 (34)

The path *P* is the collection of all $u_t, t \in [0, 1]$. The velocity η and accelleration ξ along the path are

$$\eta_t = \frac{du_t}{dt}, \quad \xi_t = \frac{d^2 u_t}{dt^2}.$$
(35)

By substituting *b* by b_t in (9) and derivating one and two times one finds easily that (we drop from now on the substript *t*) η and ξ are given by:

$$\eta \in V, \int_{\Omega} \frac{1}{b} \nabla \eta \cdot \nabla v = \int_{\Omega} \frac{c}{b^2} \nabla u \cdot \nabla v \quad \forall v \in V,$$
(36)

$$\xi \in V, \int_{\Omega} \frac{1}{b} \nabla \xi \cdot \nabla v = 2 \int_{\Omega} \left(\frac{c}{b^2} \nabla \eta - \frac{c^2}{b^3} \nabla u \right) \cdot \nabla v \quad \forall v \in V,$$
(37)

where

$$c = b_1 - b_0$$
 (38)

We want to see wether we can find α_m and R such that, for all $t \in [0, 1]$:

$$|\nabla \eta|_{L^2} \ge \alpha_m |c|_{L^2} \quad \text{with } \alpha_m > 0, \tag{39}$$

$$|\nabla\xi|_{L^2} \le \frac{1}{R} |\nabla\eta|_{L^2}^2 \quad \text{with } R > 0, \tag{40}$$

where α_m and *R* are as independent as possible of *t* and b_0, b_1 in *D*.

The geometrical interpretation of (39) (40) is as follows (Chavent and Kunisch, 96)

the constant α_m is a lower bound to the sensitivity of the b→u(b) mapping at b = b_t in the direction c of the parameter space (a sort of "guaranteed sensitivity" at b in direction c). In the finite dimensional case α_m is nothing but a lower bound to the singular values of linearized forward map.

As it will turn out in the following estimations that the "relative perturbation"

$$e = c/b \tag{41}$$

rather than the perturbation $c = b_1 - b_0$ appears in the formula, it will be convenient to replace (39) by the equivalent inequality:

$$|\nabla \eta|_{L^2} \ge \tilde{\alpha}_m |e|_{L^2} \quad \text{with } \tilde{\alpha}_m > 0. \tag{42}$$

• the constant *R* is a lower bound to the radii of curvature along *P*. It becomes infinite when *P* is a segment. Notice that (40) can hold even if (39) or (42) don't.

We search first for a lower bound to $|\nabla \eta|$.

Proposition 5.1. *for any* $b_0, b_1 \in D$ *and* $t \in [0, 1]$ *one has*

$$\left|\frac{1}{b}\nabla\eta\right| \ge (1-\mu^2)^{1/2} |d\nabla u| \tag{43}$$

where

$$\begin{cases} b = b_t = (1-t)b_o + tb_1 \\ d = \frac{c}{b^2} = \frac{b_1 - b_0}{b^2} \\ \mu = C_W(\Omega) \frac{|\nabla \wedge d . \nabla u|}{|d\nabla u|}. \end{cases}$$
(44)

Proof. inequality (43) is obtained immediately by passing to the limit after dividing by dt in inequality (30) for $b_0 = b_t, b_1 = b_{t+dt}$ when dt goes to zero.

In general, there are stagnation points inside Ω , where $|\nabla u| = 0$, in which case there is no hope to obtain a strictly positive guaranteed sensitivity $\tilde{\alpha}_m$ for e = c/b in L^2 of the whole Ω :

Corollary 5.1. Let $\tilde{\Omega} \subset \Omega$ be such that

$$|\nabla u(x)| \ge m > 0 \quad \forall x \in \tilde{\Omega}$$
(45)

Then

$$|\nabla \eta|_{L^2(\Omega)} \ge \frac{1}{m} \frac{b_m}{b_M} (1 - \mu^2)^{\frac{1}{2}} |e|_{L^2(\tilde{\Omega})}$$
(46)

Notice that one can have $\tilde{\Omega} = \Omega$ in certain circumstances, for example when f = 0 with a single well γ .

Using the equations (36) and (37) defining the velocity η and accelleration ξ , together with the grad / rot decomposition of proposition 3.1, one can obtain the following upper bound to $|\nabla \xi|$:

Proposition 5.2. For any $c = b_1 - b_0$ with $b_0, b_1 \in D$ and $t \in [0, 1]$ one has,

$$|\nabla\xi| \le 2c_{\infty} \frac{b_M}{b_m} |\nabla \wedge e| |\nabla \wedge e \cdot \nabla u| \tag{47}$$

where c_{∞} is the continuity constant of the $g \in L^{2}(\Omega) \mapsto w \in L^{\infty}(\Omega)$ mapping defined in (17).

We see from (47) that the second derivative ξ , and hence the curvature, is zero as soon as $\nabla e = \nabla \left\{ \frac{c}{b} \right\}$ is colinear to ∇u every where in Ω . It is hence the (relative) variation of the diffusion coefficient perpendicular to the flow lines which is responsible for the non-linearity in our case.

Combining proposition 5.1 and 5.2 gives immediately an estimation of the curvature 1/R of the path at $b = b_t$:

Corollary 5.2. For any $b_0, b_1 \in D$ and $t \in [0, 1]$, the curvature 1/R of P in $H^1(\Omega)$ at $b = b_t$ satisfies:

$$1/R \le 2c_{\infty} \left(\frac{b_M}{b_m}\right)^3 \frac{\mathbf{v}^2}{1-\mu^2} \tag{48}$$

where:

$$\mu = C_w(\Omega) \frac{|\nabla \wedge d.\nabla u|}{|d\nabla u|} \quad (d = c/b^2)$$
(49)

$$v^{2} = \frac{|\nabla \wedge e.\nabla u| |\nabla \wedge e|}{|e\nabla u|^{2}} \quad (e = c/b)$$
(50)

In order to see how the sensitivity $\tilde{\alpha}_m$ and the curvature 1/R are related to

$$\chi = |\nabla e| / |e| \tag{51}$$

we consider the simple case where f = 0 and Ω contains only one well, so that

$$\exists m > 0: \quad |\nabla u(x)| \ge m > 0 \quad \forall x \in \Omega, \tag{52}$$

and replace D by the smaller set

$$D_{\infty} = \{ b \in D | |\nabla b(m)| \le b_{\infty} \quad \forall x \in \Omega \}$$
(53)

with b_{∞} still to be chosen.

We search now for the constant $\tilde{\alpha}_m$ in the inverse stability estimate (42) and for the upper bound 1/R to the curvature in the inequality (40).

We notice that (43) is written in term of $\nabla d = \nabla (c/b^2)$, whereas it is $\nabla \wedge e$, which has the same norm as $\nabla e = \nabla (c/b)$, which appears in (47). Using the fact that *b* is in the set D_{∞} defined by (53), one checks easily that:

$$\frac{\nabla d}{|d|} \le \frac{b_M}{b_m} \frac{|\nabla e|}{|e|} + \frac{b_\infty}{b_m}.$$
(54)

Hence if the upperbound b_{∞} to $|\nabla b(x)|$ in D_{∞} is chosen such that

$$C_W(\Omega)\frac{M}{m}\frac{b_{\infty}}{b_m} \le \frac{1}{2},\tag{55}$$

then

$$1 - C_W(\Omega)^2 \frac{|\nabla \wedge d.\nabla u|^2}{|d\nabla u|^2} \ge \frac{1}{2} \left(1 - 4C_W(\Omega)^2 \frac{M^2}{m^2} \left(\frac{b_M}{b_m}\right)^2 \chi^2 \right).$$
(56)

Combining (43) and (47) with (56) gives the expected result:

Proposition 5.3. Suppose that the admissible set D_{∞} satisfies (55). Let $b_0, b_1 \in D_{\infty}$ and $t \in [01]$ be given, and define

$$K = 2C_W(\Omega) \frac{M}{m} \frac{b_M}{b_m}$$
(57)

where $e = c/b = (b_1 - b_0)/b$ with $b = b_t$. Then, under the condition

$$K\chi < 1$$
 (58)

inequalities (42), and (40) hold with:

$$\tilde{\alpha}_m = \frac{\sqrt{2}}{2} m \frac{b_m}{b_M} (1 - K^2 \chi^2)^{1/2}$$
(59)

$$1/R = 2\sqrt{2}c_{\infty}\frac{M}{m^2} \left(\frac{b_M}{b_m}\right)^3 \frac{\chi^2}{(1 - K^2\chi^2)}$$
(60)

This proposition shows that the curvature 1/R of P in $H^1(\Omega)$ will remain finite if e = c/b is "not too far from a constant" in the sense that $\chi = \frac{|\nabla e|}{|e|}$ is small. The curvature is exactely zero when $\chi = 0$, ie when c/b is a constant (which is not very surprising...). It increases when χ departs from zero, and the upper bound 1/R blows up to infinity when χ approaches 1/K. At the same time, the "minimum guaranted sensitivity" $\tilde{\alpha}_m$ tends to zero.

These estimates confirm the numerical observations made in (Liu, 93) (Grimstad and Mannseth, 99) that directions of high non linearity seem to coincide with directions with low sensitivity.

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