# VARIATION BOUNDARY INTEGRAL EQUATION FOR FLAW SHAPE IDENTIFICATION 

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#### Abstract

In this communication a Variation Boundary Integral Equation (BIE) for the solution of identification inverse problems is presented. This equation relates the variation of the fields along the boundary with the variation of the geometry of a flaw, whose position and shape are unknown beforehand. The Variation BIE is obtained linearizing the difference between the standard BIE for the actual configuration (actual flaw) and the standard BIE for the assumed configuration. The resulting Variation BIE has not been completely derived before, to the authors knowledge. The solution of the ensuing Variation BIE is tackled by a procedure that avoids altogether the solution of a nonlinear minimization problem. The variations of the design variables (geometry) have been written in terms of a virtual strain field applied to the flaw. This approach can be applied to flaws of any shape and location. Several numerical applications are solved with the proposed formulation.


## INTRODUCTION

The study of inverse problems in engineering has become and active area of research in the last two decades (Tanaka and Dulikravich, 1998; Delaunay and Jarny, 1996; Zabaras, Woodbury and Raynaud, 1993). There are many important topics in engineering which involve the solution of this kind of problems: non-destructive evaluation of materials and structures, character-

[^0]ization of material properties, medical imaging or even diagnosis, etc.

In this paper we consider the identification inverse problem in the case of an acoustic field over a two-dimensional domain. The aim in the identification inverse problem is to compute an inaccessible part of the boundary of the domain, usually an internal flaw, using experimental data as additional information. Since the objective is to find part of the boundary, an approach based on Boundary Integral Equations appears as a very sensible alternative.

The derivation of the gradient for such problems has been tackle by several researches by either the adjoint variable approach or direct differentiation. The first method has been employed by several authors in different applications, i.e. Aithal and Saigal (1995) for two-dimensional thermal problems; Bonnet (1995a) for 3D inverse scattering problems by hard and penetrable obstacles; Meric (1995) for shape optimization in potential fields. The direct differentiation approach has been extensively used as well, for regular, strongly, and even hypersingular BIE (Mellings and Alliabadi, 1993; Bonnet, 1995b; Matsumoto et al., 1993; Nishimura and Kobayashi, 1991; Kirsch, 1993).

Zeng and Saigal (1992) developed a formulation for potential fields based on variations. Tanaka and Matsuda (1989) developed a similar approach years earlier using Taylor expansions of the kernels and densities in the BIE. In these two papers, the authors propose an approach different to the minimization of a cost functional, but failed to demonstrate its reliability, due to mathematical inconsistencies (Tanaka and Matsuda 1989) or simply
because no numerical application is carried out (Zeng and Saigal, 1992). Gallego and Suarez (1999a) developed the variation Boundary Integral Equation ( $\delta$ BIE) and presented some numerical results using the non-minimization approach. Nevertheless, the $\delta$ BIE should be equivalent to the sensitivity integral equations obtained by the direct differentiation approach, and therefore, the $\delta$ BIE can be used to compute the gradient of a given cost function, although no effort has been undertaken in this direction yet.

In this communication we demonstrate that the $\delta$ BIE is a reliable method to solve identification inverse problems for acoustic fields. First the $\delta$ BIE is reviewed and a numerical procedure for its solution by collocation boundary element discussed. The virtual strain field is introduced for the geometrical parameter representation, which allows for greater flexibility for the shape of the assumed flaw. An iterative procedure for the solution of the identification inverse problem is proposed, and finally, some numerical applications are presented.

## INVERSE PROBLEM FOR LINEAR ACOUSTIC MEDIA: BASIC EQUATIONS

In this section the basic equations for the solution of harmonic wave propagation problems in a linear acoustic media are reviewed. In the last paragraph the identification inverse problem is briefly described.

## Differential equation statement

Consider the well known problem of an acoustic field,

$$
\begin{equation*}
\nabla^{2} u(\mathbf{x})+k^{2} u(\mathbf{x})=0 ; \quad \mathbf{x} \in \Omega \tag{1}
\end{equation*}
$$

subject to essential and/or natural boundary conditions,

$$
\begin{array}{ll}
u(\mathbf{x})=\bar{u} ; & \mathbf{x} \in \Gamma_{u} \\
q(\mathbf{x})=\bar{q} ; & \mathbf{x} \in \Gamma_{q}
\end{array}
$$

where, $u(\mathbf{x})$ is the acoustic field in $\Omega ; q(\mathbf{x})=\partial u / \partial \mathbf{n}$ is the flux at a point $\mathbf{x}$ on the boundary $\Gamma$ whose outward normal is $\mathbf{n}(\mathbf{x})$; $k=\frac{\omega}{c}$ is the wave number, $\omega$ the frequency of the wave and $c$ its velocity; $\bar{u}$ and $\bar{q}$ represent known values of the field and flux on $\Gamma_{u}$ and $\Gamma_{q}$ respectively, where $\Gamma_{u} \cup \Gamma_{q}=\Gamma$ and $\Gamma_{u} \cap \Gamma_{q}=\emptyset$.

To solve the problem stated in equation (1) one needs to know (Kubo, 1988): the domain $\Omega$ and its boundary $\Gamma$, the differential operator ( $\nabla^{2}+k^{2}$ in this case), boundary conditions ( $\bar{u}$ and $\bar{q}$, and their supports $\Gamma_{u}$ and $\Gamma_{q}$ ), and material properties $(c)$. If one or several of these items are not completely known the problem stated in equation (1) will not be well-posed and will not have solution or if any, will not be unique. An inverse problem can therefore be stated whose objective is to find the missing information, using some additional data. Depending on the
missing information different kinds of inverse problems can be stated. In this paper we deal with the so called identification inverse problem whose objective is to find part of the domain or its boundary.

## Integral equation statement

In the identification problem part of the boundary, say $\tilde{\Gamma}_{h}$, is the main unknown of the problem. Therefore the statement of the problem in terms of Boundary Integral Equations (BIE) appears as the most promising approach.

The acoustic problem, stated in differential form in equation (1), can be written in terms of BIE (Dominguez, 1993) by the equation,

$$
\begin{equation*}
c(\mathbf{x}) u(\mathbf{x})=\int_{\Gamma}\left\{u^{*}(\mathbf{y} ; \mathbf{x}) q(\mathbf{y})-q^{*}(\mathbf{y} ; \mathbf{x}) u(\mathbf{y})\right\} d \Gamma(\mathbf{y}) \tag{2}
\end{equation*}
$$

where $c(\mathbf{x})$, called the free term, is 0 if $\mathbf{x} \notin \Omega \cup \Gamma, 1$ if $\mathbf{x} \in \Omega$ and $\theta / 2 \pi$ if $\mathbf{x} \in \Gamma$, where $\theta$ is the interior angle between the left and right tangents to the boundary at the point $\mathbf{x} ; u^{*}(\mathbf{y} ; \mathbf{x})$ and $q^{*}(\mathbf{y} ; \mathbf{x})$ are given by

$$
\begin{align*}
& u^{*}(\mathbf{y} ; \mathbf{x})=\frac{1}{2 \pi} K_{0}(z)  \tag{3}\\
& q^{*}(\mathbf{y} ; \mathbf{x})=-\frac{1}{2 \pi r} z K_{1}(z) \mathbf{p} \cdot \mathbf{n} \tag{4}
\end{align*}
$$

where $z=i \omega r / c$, and represent the fundamental solution and its flux for the Helmholtz equation; $r=|\mathbf{y}-\mathbf{x}|$ is the distance between the collocation point $\mathbf{x}$ and the integration or observation point $\mathbf{y}$; $i$ is the imaginary unit; $K_{m}($.$) is the modified Bessel func-$ tion of second kind and $m^{t h}$ order; $\boldsymbol{\rho}$ is a unit vector in the direction of $\mathbf{y}-\mathbf{x}$; and $\mathbf{n}$ is the outward normal to the boundary at the observation point.

This boundary integral equation (BIE) can be used to solve the direct problem stated in equation (1) if all the needed information is provided. If part of the boundary is not known, this BIE can be used to derive a new $\delta$ BIE which relates the value of the variables on the boundary, their variations and the geometrical variation from an assumed shape and position of the missing part of the boundary, as shown in the next section.

## Identification inverse problem

In the sequel we will denote with a tilde () the variables and parameters in the real configuration, if they are different to their counterparts in the assumed one.

In the identification problem a portion of the boundary, termed $\tilde{\Gamma}_{h}$, is not known. Usually $\tilde{\Gamma}_{h}$ represents the boundary of an interior flaw whose shape and location is sought. In order to find this flaw, additional data has to be provided, besides the
known boundary conditions. For example, experimental measurements may be available at a set of points on $\Gamma_{c}$, the known and accessible portion of the boundary,

$$
\begin{aligned}
u\left(\mathbf{x}_{\alpha}\right) & =\bar{u}\left(\mathbf{x}_{\alpha}\right), & & \mathbf{x}_{\alpha} \in \Gamma_{c} \\
q\left(\mathbf{x}_{\beta}\right) & =\bar{q}\left(\mathbf{x}_{\beta}\right), & & \mathbf{x}_{\beta} \in \Gamma_{c}
\end{aligned}
$$

where $\alpha=1, \ldots M_{u}$ and $\beta=1, \ldots M_{q}$ and therefore, $M=M_{u}+$ $M_{q}$ supplementary values are known. In addition, measurements at points inside the domain $\Omega$ can be provided as well. This additional data will provide information to estimate the shape and position of the unknown flaw.

## BOUNDARY INTEGRAL EQUATION FOR THE IDENTIFICATION INVERSE PROBLEM

By linearizing equation (2) with respect to small variations of the geometry, a new BIE is obtained which relates the variation of the variables along the boundary with the transformation of the geometry. In the next paragraphs the variations are defined, and then the Variation BIE is presented. Finally this BIE is established for a point on the boundary.

## Variation of the geometry and boundary variables

To transform the assumed domain $\Omega$ to the actual domain $\tilde{\Omega}$ a point $\mathbf{x}$ is applied to a new point $\tilde{\mathbf{x}}=\mathbf{x}+\delta \mathbf{x}$, where $\delta \mathbf{x}$ is the variation of the geometry. It has to be emphasized that the whole domain is distorted in order to change the shape and position of the flaw from its assumed location to the actual one (Figure 1), and not only the points on the boundary of the flaw.

The linearized integral equation will be written in terms of the difference of the potential and flux between the actual and the assumed domain. Then, the variation of the potential in the assumed configuration is defined as,

$$
\begin{equation*}
\delta u(\mathbf{x})=u\left(\mathbf{x}, \tilde{\Gamma}_{h}\right)-u\left(\mathbf{x}, \Gamma_{h}\right) \tag{5}
\end{equation*}
$$

Therefore, $\delta u$ represents the difference in the potential at a given point $\mathbf{x}$ due to the variation of the boundary of the domain. To define $\delta q$ extra care has to be exercised since the flux is defined at the boundary of the domain, and this boundary changes when the geometry is distorted. Taking into account that $q(\mathbf{x})=\partial u / \partial \mathbf{n}=$ $\nabla u(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})$, the following definition has been adopted,

$$
\begin{equation*}
\delta q(\mathbf{x})=\left\{\nabla u\left(\mathbf{x}, \tilde{\Gamma}_{h}\right)-\nabla u\left(\mathbf{x}, \Gamma_{h}\right)\right\} \cdot \mathbf{n}(\mathbf{x})=\nabla \delta u(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \tag{6}
\end{equation*}
$$

These are local variations since they are computed for a fixed point, $\mathbf{x}$. Alternative definitions would be the material variations


Figure 1. CONTINUOUS TRANSFORMATION FROM THE ASSUMED DOMAIN, $\Omega$ TO THE REAL ONE, $\tilde{\Omega}$
given by,

$$
\begin{equation*}
\delta u(\mathbf{x})=u\left(\tilde{\mathbf{x}}, \tilde{\Gamma}_{h}\right)-u\left(\mathbf{x}, \Gamma_{h}\right) \tag{7}
\end{equation*}
$$

and,

$$
\begin{equation*}
\delta q(\mathbf{x})=q\left(\tilde{\mathbf{x}}, \tilde{\Gamma}_{h}\right)-q\left(\mathbf{x}, \Gamma_{h}\right)=\nabla u\left(\mathbf{x}, \tilde{\Gamma}_{h}\right) \cdot \tilde{\mathbf{n}}(\tilde{\mathbf{x}})-\nabla u\left(\mathbf{x}, \Gamma_{h}\right) \cdot \mathbf{n}(\mathbf{x}) \tag{8}
\end{equation*}
$$

Both definitions for the variations are possible, but the first alternative has been used in this work. The consequence of using the second definitions for the variations in the ensuing BIE are being currently explored.

## Variation integral equation

Equation (2) can be written for an interior point $\mathbf{x}$ in the assumed domain,

$$
\begin{align*}
u\left(\mathbf{x}, \Gamma_{h}\right)= & \int_{\Gamma_{c}}\left\{u^{*}(\mathbf{y} ; \mathbf{x}) q\left(\mathbf{y}, \Gamma_{h}\right)-q^{*}(\mathbf{y} ; \mathbf{x}) u\left(\mathbf{y}, \Gamma_{h}\right)\right\} d \Gamma(\mathbf{y})+ \\
& \int_{\Gamma_{h}}\left\{u^{*}(\mathbf{y} ; \mathbf{x}) q\left(\mathbf{y}, \Gamma_{h}\right)-q^{*}(\mathbf{y} ; \mathbf{x}) u\left(\mathbf{y}, \Gamma_{h}\right)\right\} d \Gamma(\mathbf{y}) \tag{9}
\end{align*}
$$

and the same equation can be set at the corresponding point $\tilde{\mathbf{x}}$ in the actual domain,

$$
u\left(\tilde{\mathbf{x}}, \tilde{\Gamma}_{h}\right)=\int_{\Gamma_{c}}\left(u^{*}(\mathbf{y} ; \tilde{\mathbf{x}}) q\left(\mathbf{y}, \tilde{\Gamma}_{h}\right)-q^{*}(\mathbf{y} ; \tilde{\mathbf{x}}) u\left(\mathbf{y}, \tilde{\Gamma}_{h}\right)\right) d \Gamma(\mathbf{y})+
$$

$$
\begin{equation*}
\int_{\tilde{\Gamma}_{h}}\left(u^{*}(\tilde{\mathbf{y}} ; \tilde{\mathbf{x}}) q\left(\tilde{\mathbf{y}}, \tilde{\Gamma}_{h}\right)-q^{*}(\tilde{\mathbf{y}} ; \tilde{\mathbf{x}}) u\left(\tilde{\mathbf{y}}, \tilde{\Gamma}_{h}\right)\right) d \Gamma(\tilde{\mathbf{y}}) \tag{10}
\end{equation*}
$$

Computing the series expansion of the terms of equation (10) with respect to the variation of the geometry $\delta \mathbf{x}$, neglecting terms of quadratic order and higher, and calculating the difference of the ensuing expansion with the equation (9), the following Variation Boundary Integral Equation is obtained,

$$
\begin{align*}
& \delta u(\mathbf{x})+\nabla u\left(\mathbf{x}, \Gamma_{h}\right) \cdot \delta \mathbf{x}= \\
& \quad \int_{\Gamma}\left\{u^{*}(\mathbf{y} ; \mathbf{x}) \delta q(\mathbf{y})-q^{*}(\mathbf{y} ; \mathbf{x}) \delta u(\mathbf{y})\right\} d \Gamma(\mathbf{y})  \tag{11}\\
& -\int_{\Gamma_{c}}\left\{\nabla u^{*}(\mathbf{y} ; \mathbf{x}) q\left(\mathbf{y}, \Gamma_{h}\right)-\nabla q^{*}(\mathbf{y} ; \mathbf{x}) u\left(\mathbf{y}, \Gamma_{h}\right)\right\} \cdot \delta \mathbf{x} d \Gamma(\mathbf{y}) \\
& +\int_{\Gamma_{h}}\left\{\left[\nabla u^{*}(\mathbf{y} ; \mathbf{x}) q\left(\mathbf{y}, \Gamma_{h}\right)-\nabla q^{*}(\mathbf{y} ; \mathbf{x}) u\left(\mathbf{y}, \Gamma_{h}\right)\right] \cdot(\delta \mathbf{y}-\delta \mathbf{x})+\right. \\
& \quad\left[u^{*}(\mathbf{y} ; \mathbf{x}) \nabla q\left(\mathbf{y}, \Gamma_{h}\right)-q^{*}(\mathbf{y} ; \mathbf{x}) \nabla u\left(\mathbf{y}, \Gamma_{h}\right)\right] \cdot \delta \mathbf{y}+ \\
& \left.\quad\left[u^{*}(\mathbf{y} ; \mathbf{x}) \nabla u\left(\mathbf{y}, \Gamma_{h}\right)-\nabla u^{*}(\mathbf{y} ; \mathbf{x}) u\left(\mathbf{y}, \Gamma_{h}\right)\right] \cdot \delta \mathbf{m}(\mathbf{y})\right\} d \Gamma(\mathbf{y})
\end{align*}
$$

This BIE relates the variation of the potential and the geometry at a point $\mathbf{x} \in \Omega$, with the variation of the potential, flux and geometry along the boundary of the domain. The potential and flux of the primary problem on the assumed configuration, and their gradients appear in the equation as well, but they can be computed solving the direct problem.

This integral equation would be much more useful if we are able to write it for a point $\xi \in \Gamma$, since, in such case, only quantities along the boundary will be involved. A careful limiting process (see Guiggiani, 1992 and Gallego et al., 1996) such that $\mathbf{x} \rightarrow \boldsymbol{\xi} \in \Gamma$ leads to,

$$
\begin{align*}
& c(\xi)(\delta u(\xi)+\nabla u(\xi) \cdot \delta \xi)+\mathbf{b}(\xi): u(\xi) \nabla \delta \xi=  \tag{12}\\
& \int_{\Gamma}\left\{u^{*} \delta q-q^{*} \delta u\right\} d \Gamma-\int_{\Gamma}\left\{\nabla u^{*} q-\nabla q^{*} u\right\} \cdot \delta \xi d \Gamma+ \\
& \int_{\Gamma_{h}}\left\{\left[\nabla u^{*} q-\nabla q^{*} u\right] \cdot \delta \mathbf{y}+\left[u^{*} \nabla q-q^{*} \nabla u\right] \cdot \delta \mathbf{y}+\right. \\
& \left.\quad\left[u^{*} \nabla u-\nabla u^{*} u\right] \cdot \delta \mathbf{m}\right\} d \Gamma
\end{align*}
$$

which is the Variation BIE for a point $\xi \in \Gamma$, where $c(\xi)=$ $\theta / 2 \pi$ and

$$
\mathbf{b}(\xi)=\frac{1}{4 \pi}\left(\begin{array}{cc}
\sin 2 \theta_{2}-\sin 2 \theta_{1} & \cos 2 \theta_{1}-\cos 2 \theta_{2}  \tag{13}\\
\cos 2 \theta_{1}-\cos 2 \theta_{2} & \sin 2 \theta_{1}-\sin 2 \theta_{2}
\end{array}\right)
$$

This Variation Boundary Integral Equation ( $\delta \mathrm{BIE}$ in the sequel) is valid for any point $\xi \in \Gamma$, both on the known boundary $\Gamma_{c}$ and the unknown boundary $\Gamma_{h}$.

The kernels in this equation are as the ones in the primary BIE (2) plus their gradients,
$\nabla u^{*}(\mathbf{y} ; \mathbf{x})=-\frac{\boldsymbol{\rho}}{2 \pi r} z K_{1}(z)$
$\boldsymbol{\nabla} q^{*}(\mathbf{y} ; \mathbf{x})=-\frac{1}{2 \pi r^{2}}\left\{z K_{1}(z) \mathbf{n}-\left(z K_{1}(z)-\frac{1}{2} z^{2} K_{0}(z)\right) 2 \boldsymbol{\rho} \boldsymbol{\rho} \cdot \mathbf{n}\right\}$

## Dynamic kernels expansion

In order to asses the order of the boundary singularities involved in the $\delta$ BIE, the kernel series expansions as $r \rightarrow 0$ have been computed.

The modified Bessel functions of the second kind can be expanded as,

$$
\begin{align*}
& K_{0}(z)=-\ln z-\gamma+O(z)  \tag{16}\\
& K_{1}(z)=\frac{1}{z}+\frac{z}{2}\left(\ln \frac{z}{2}+\gamma-\frac{1}{2}\right)+O\left(z^{2}\right) \tag{17}
\end{align*}
$$

Then, it is easily shown that,

$$
\begin{align*}
u^{*}(\mathbf{y} ; \mathbf{x}) & =U(\mathbf{y} ; \mathbf{x})+O(z)  \tag{18}\\
q^{*}(\mathbf{y} ; \mathbf{x}) & =Q(\mathbf{y} ; \mathbf{x})+O(z \ln z) \tag{19}
\end{align*}
$$

where

$$
\begin{align*}
& U(\mathbf{y} ; \mathbf{x})=-\frac{1}{2 \pi} \ln r  \tag{20}\\
& Q(\mathbf{y} ; \mathbf{x})=-\frac{1}{2 \pi r} \mathbf{p} \cdot \mathbf{n} \tag{21}
\end{align*}
$$

are the kernels of the static problem (Laplace equation). Likewise,

$$
\begin{align*}
& \nabla u^{*}(\mathbf{y} ; \mathbf{x})=\nabla U(\mathbf{y} ; \mathbf{x})+O(z \ln z)  \tag{22}\\
& \nabla q^{*}(\mathbf{y} ; \mathbf{x})=\nabla Q(\mathbf{y} ; \mathbf{x})+O(\ln z) \tag{23}
\end{align*}
$$

where

$$
\begin{align*}
& \nabla U(\mathbf{y} ; \mathbf{x})=-\frac{\boldsymbol{\rho}}{2 \pi r}  \tag{24}\\
& \nabla Q(\mathbf{y} ; \mathbf{x})=-\frac{1}{2 \pi r^{2}}(\mathbf{n}-2 \boldsymbol{\rho} \boldsymbol{\rho} \cdot \mathbf{n}) \tag{25}
\end{align*}
$$

As expected, the order of the singularity of the kernels is as in the static counterpart of the $\delta$ BIE (Gallego and Suarez, 1999a).

## NUMERICAL SOLUTION OF THE SBIE

The order of the singularities involved leads to the following conclusions,

1. The $\delta$ BIE is one order of singularity higher that the original BIE, due to the term in $\nabla u^{*} u \cdot \delta \mathbf{m}$, which is $O\left(r^{-1}\right)$.
2. The equation is not hypersingular, in spite of the term $\nabla q^{*}$ since it is multiplied by the function $(\delta \mathbf{y}-\delta \xi)$ which vanishes at $r=0$
3. The variables $u(\mathbf{y})$ and $q(\mathbf{y})$ have to be $C^{0}$ at $\xi$, as in the standard BIE. In addition, the limiting process (Gallego and Suarez, 1999a) shows that $\nabla u \cdot \delta \mathbf{y}$ has to be $C^{0}$ at $\xi$ as well.
4. Finally the most restrictive continuity condition is that $\delta \mathbf{y}$ has to be $C^{1}$ at the collocation point.

The increase in the order of the singularity is not critical since the primary BIE is weakly singular, and therefore the variation BIE is strongly singular, but not hypersingular, and can be easily handle.

The last two remarks, however, are specially important since these continuity restrictions apply to the approximate variables as well. The $C^{1}$ continuity for the $\delta \mathbf{y}$ is difficult to fulfill at the nodes on the contour (3D) or the ends (2D) of the elements, and therefore a special approach has to be devised. The alternatives to fulfill this continuity condition can be reduced to the following: $C^{1}$ elements, discontinuous elements, independent interpolations for the variable and its gradient, and Multiple Collocation Approach. This last method, which is the one employed in this work, consists in collocating the $\delta$ BIE at nodes inside the elements but without changing the interpolation of the variables, therefore maintaining the same number of unknowns as in the standard continuous elements. Gallego et al. (1996) proposed this approach for 2D fracture dynamic problems, and Dominguez et al. (1999) extended it to 3D elastostatic fracture cases. This alternative is easy to implement in a conventional boundary element code.

## Parametric variation of the geometry

The aim in the solution of the identification inverse problem is to find the shape and position of an unknown flaw. A problem therefore arises which is how to parameterize these shape and position in order to define the geometry of the flaw with a finite, small number of geometrical parameters or design variables. For the present procedure what has to be written in terms of a finite number of parameters is the vector $\delta \mathbf{y}$, along the assumed boundary $\Gamma_{h}$. An obvious choice would be to define the vector $\delta \mathbf{y}$ at every node in the discretization of the flaw, but this will lead to a big number of unknowns. Since the identification inverse problem is ill-conditioned, a big number of geometrical parameters could lead to unstable and/or non-convergent results.

In the present communication we use the parameterization already tested by the authors for the static problem (Gallego
and Suarez 1999b): it is assumed that the flaw is modified as if subject to a virtual strain field. The flaw suffers a uniform displacement $\left(\delta y_{1}^{o}, \delta y_{2}^{o}\right)$, a rotation around its centroid $\delta \omega^{o}$ and a distortion and deformation field given by the parameters $\left(\delta \varepsilon_{11}^{o}, \delta \varepsilon_{22}^{o}, \delta \varepsilon_{12}^{o}\right)$.

It can be readily shown that the displacement vector $\left(u_{1}, u_{2}\right)$ at every point of the cavity $\left(y_{1}, y_{2}\right)$, due to a uniform virtual strain field can be computed by the formula:
where $\left[y_{i}\right]=y_{i}-y_{i}^{o}$, and $\mathbf{y}^{o}$ is the centroid of the assumed cavity.
An equivalent parameterization can be established using the natural decomposition of the strain tensor in its isotropic and deviatoric parts. This approach can be extended in order to allow for a greater number of parameters (see Gallego and Suarez 1999b).

## Boundary Element discretization

The numerical solution of equation (12) can be tackled using Boundary Element techniques. In this paper quadratic isoparametric elements has been employed. Thus, a scalar function, say $p(\mathbf{y})$ will be interpolated at a given element $e$ as,

$$
\begin{equation*}
p(\mathbf{y})=\phi_{1} p_{e}^{1}+\phi_{2} p_{e}^{1}+\phi_{3} p_{e}^{1}=\boldsymbol{\phi} \mathbf{p}_{e} \tag{27}
\end{equation*}
$$

where $\boldsymbol{\phi}=\left(\phi_{1}, \phi_{2}, \phi_{3}\right)$ and $\mathbf{p}_{e}=\left(p_{e}^{1}, p_{e}^{2}, p_{e}^{3}\right)^{T}$. $\phi_{i}$ are the standard quadratic shape functions, defined in a reference element whose natural coordinate is $\xi \in[-1,1]$, while $p_{e}^{i}$ are the values of the variable $p(\mathbf{y})$ at the nodes of the element. The same kind of interpolation can be used for all the variables in the $\delta$ BIE (12). Using the equation (26) to compute the variation of the geometry at the three nodes of an element, the following matrix relationship is obtained,

$$
\begin{equation*}
\delta \mathbf{y}_{j}=\mathbf{P}_{j} \delta \mathbf{v} \tag{28}
\end{equation*}
$$

where $\delta \mathbf{v}=\left(\delta y_{1}^{o}, \delta y_{2}^{o}, \delta \omega^{o}, \delta \varepsilon_{11}^{o}, \delta \varepsilon_{22}^{o}, \delta \varepsilon_{12}^{o}\right)^{T}$ is the geometrical parameter vector, and, $\mathbf{P}_{j}$, the parametric matrix for the element $j$, which depends on the values of the increments $\left|\left[y_{1}\right]\right|$ and $\left.\| y_{2}\right]$ for the three nodes of the element.

Using the interpolation (27) for all the variables in the $\delta$ BIE and (28) for the variation of the geometry, after discretization the
equation (12) can be written as,

$$
\begin{equation*}
\sum_{j=1}^{N_{e}} \mathbf{h}_{i j} \delta \mathbf{u}_{j}=\sum_{j=1}^{N_{e}} \mathbf{g}_{i j} \delta \mathbf{q}_{j}+\Delta_{i} \delta \mathbf{v} \tag{29}
\end{equation*}
$$

where,

$$
\begin{align*}
\mathbf{h}_{i j} & =\frac{1}{2} \boldsymbol{\phi}\left(\xi_{i}\right) \delta_{i j}+\int_{\Gamma_{j}} q^{*} \phi d \Gamma_{j}  \tag{30}\\
\mathbf{g}_{i j} & =\int_{\Gamma_{j}} u^{*} \phi d \Gamma_{j}  \tag{31}\\
\boldsymbol{\Delta}_{i} & =\sum_{j=1}^{N_{e}}\left(\mathbf{d}_{i j}^{a} \mathbf{P}_{j}\right)-\sum_{j=1}^{N_{e}}\left(\mathbf{d}_{i j}^{b}\right) \boldsymbol{\Phi}\left(\xi_{i}\right) \mathbf{P}_{i}-\frac{1}{2} \nabla u\left(\xi_{i}\right) \cdot \boldsymbol{\phi}(\xi) \mathbf{P}_{i} \tag{32}
\end{align*}
$$

The terms $\mathbf{d}_{i j}^{a}$ and $\mathbf{d}_{i j}^{b}$ are given by the element integrals,

$$
\begin{align*}
& \mathbf{d}_{i j}^{a}=\int_{\Gamma_{j}}\left\{\left(\nabla u^{*} q-\nabla q^{*} u+u^{*} \nabla q-q^{*} \nabla u\right) \cdot \Phi+\right. \\
& \left.\quad\left(u^{*} \nabla u-\nabla u^{*} u\right) \cdot \Psi\right\} d \Gamma_{j}  \tag{33}\\
& \mathbf{d}_{i j}^{b}=\int_{\Gamma_{j}}\left\{\nabla u^{*} q-\nabla q^{*} u\right\} d \Gamma_{j} \tag{34}
\end{align*}
$$

where $\Phi$ and $\Psi$ are matrices which collect the interpolation functions and their derivatives, respectively.

The former discrete equation (29) can be established in a set of collocation nodes. In this work we have adopted the following scheme: for the elements on the known part of the boundary $\Gamma_{c}$, standard collocation is performed since $\delta \mathbf{y}=0$ in these elements, and therefore, the continuity requirements are obviously fulfilled; for each element on the unknown part, the cavity, three equations are established: one for the central node, and one at a point close to each end of the element. The equation at each end is simply added to the corresponding equation at the same end in the contiguous element. The final discrete system can be written as,

$$
\begin{equation*}
\mathbf{H} \delta \mathbf{u}=\mathbf{G} \delta \mathbf{q}+\Delta \delta \mathbf{v} \tag{35}
\end{equation*}
$$

where $\mathbf{H}$ and $\mathbf{G}$ are $N \times N$ and $N \times N_{e}$ matrices respectively, $N$ being the number of interpolation nodes, while $\Delta$ is a $N \times 6$ matrix.

## Iterative solution of the inverse problem

In this section, the numerical solution of the set of integral equations comprised by the BIE of the direct problem (2) and the $\delta$ BIE of the inverse problem (12) is summarized.

The discretization of the primary BIE leads to the well known set of algebraical equations,

$$
\begin{equation*}
\mathbf{H u}=\mathbf{G q} \tag{36}
\end{equation*}
$$

where the vectors $\mathbf{u}=\left(u^{1}, u^{2}, \ldots, u^{N}\right)$ and $\mathbf{q}=\left(q^{1}, q^{2}, \ldots, q^{N}\right)$ collect the potential and flux at the interpolation nodes. After the application of the primary boundary conditions to the former set, a square system of equations is obtained whose solution completely determined the vectors $\mathbf{u}$ and $\mathbf{q}$.

On the other hand, the discretization of the inverse $\delta$ BIE leads to the following set of equations 35 ,

$$
\begin{equation*}
\mathbf{H} \delta \mathbf{u}=\mathbf{G} \delta \mathbf{q}+\Delta \delta \mathbf{v} \tag{37}
\end{equation*}
$$

where the vectors $\delta \mathbf{u}$ and $\delta \mathbf{q}$ collect the variations of the potential and the flux at the interpolation nodes, respectively. The variation of the geometry of the boundary $\Gamma_{h}$ is in the geometrical parameter vector $\delta \mathbf{v}$ as shown in the previous subsection. The application of the boundary conditions for the inverse problem yields,

$$
\begin{equation*}
\mathbf{H}_{\mathbf{r}} \delta \mathbf{x}=\tilde{\Delta} \delta \mathbf{v} \tag{38}
\end{equation*}
$$

where $\delta \mathbf{x}$ are the N unknown variations of the potential and/or fluxes. The right hand side matrix $\tilde{\Delta}$ stems from $\Delta$ and the boundary conditions.

The former set of N equations cannot be solved since the number of unknowns is $N+6$. To solve it, the M experimental values are taken into account. At the points on $\Gamma_{c}$ where the potential is measured, $\delta u=u\left(\mathbf{y}, \tilde{\Gamma}_{h}\right)-u\left(\mathbf{y}, \Gamma_{h}\right)$ can be computed since $u\left(\mathbf{y}, \Gamma_{h}\right)$ is known from the primary problem. Likewise at the points where the flux is measured, $\delta q$ can be computed. The number of unknowns is therefore reduced to $N+6-M$. Collecting the unknowns to the left hand side a non-square system of equations is obtained

$$
\begin{equation*}
\mathbf{A} \delta \mathbf{h}=\mathbf{c} \tag{39}
\end{equation*}
$$

The number of equations is $N$ and the number of unknowns $N-M+6$. Obviously, $M \geq 6$, i.e. the number of experimental measurements should be greater or equal to the number of geometrical parameters, in order to obtain a square or overdetermined system of equations. The solution of this overdetermined system of equations can be tackled by standard least squares techniques.

The solution of these equations yields the unknown geometrical parameters $\delta \mathbf{v}$. By equation (28) $\delta \mathbf{y}_{j}$ is computed and the flaw shape updated. The procedure is repeated after convergence is attained.


Figure 2. L-SHAPE FLAW: GEOMETRY, REAL FLAW AND BOUNDARY CONDITIONS

## NUMERICAL EXAMPLES

In this section the solution of three series of problems is presented. In all series the known geometry and the real shape and position of the flaw is the same, and the boundary conditions as well. The material wave propagation velocity is $c=0.1$

The defect is an L-shape flaw at the right upper corner of a $2 \times 2$ square. The boundary conditions and the geometry are shown in Figure 2. The flaw is 0.1 units wide and its arms are 0.3 units long. The Boundary Element model has sixteen quadratic elements for the exterior boundary and ten quadratic elements for the assumed flaw.

In the real problem, with the boundary conditions shown in Figure 2, the frequency of the excitation has been modified in a wide range. The ensuing amplification of the potential (displacement considering an antiplane problem) at the middle of the left side can be used to compute the natural frequencies of the problem. The first natural frequency is about $\omega_{n}=0.075 s^{-1}$ and it will be used as a reference.

For each series, three frequencies have been tested $\omega_{1} / \omega_{n}=$ $0.53, \omega_{2} / \omega_{n}=0.8$ and $\omega_{3} / \omega_{n}=1.47$, the first two under the first natural frequency and the third above it.

## L-shape crack: exact shape for the assumed flaw

The assumed flaw has the shape and orientation of the real flaw but it is displaced to the left lower corner of the square (see Figure 2). Depending on the frequency a number of experimental measures $M$ has been simulated for this application: $M=6$ for $\omega_{1}$ and $M=20$ for $\omega_{2}$ and $\omega_{3}$. These measures are exact in the sense that they have been computed by a direct BE code and no experimental error has been included.

The initial, intermediate and final position of the flaw are shown in Figures 3 to 5 for the frequencies $\omega_{1}$ to $\omega_{3}$ respectively.


Figure 3. L-SHAPE FLAW: FREQUENCY $\omega_{1} / \omega_{n}=0.53$ (L-SHAPE INITIAL FLAW)


Figure 4. L-SHAPE FLAW: FREQUENCY $\omega_{2} / \omega_{n}=0.8$ (L-SHAPE INITIAL FLAW)

## L-shape crack: circular initial flaw

For the second series the initially assumed flaw is a centered circle with radius 0.1 . A set of $M=20$ exact experimental data is provided, five in each side of the square. For the highest frequency, $\omega_{3}$, in order to attain convergence the solution of the overdetermined system of equations in equation (39) is solved by a weighted least square method where the weights of the equations corresponding to the experimental measures are set to 10 while the rest of the weights are kept to 1 . In this series the deformation of the flaw is restricted and it can only be displaced but not deformed.

The initial, intermediate and final position of the flaw are shown in Figures 6 to 8 for the frequencies $\omega_{1}$ to $\omega_{3}$ respectively.

## L-shape crack: circular initial flaw

This third and last series is like the foregoing, but the variation of the assumed flaw is now parameterized with the vector $\delta \mathbf{v}=\left(\delta y_{1}^{o}, \delta y_{2}^{o}, \delta \omega^{o}, \delta \varepsilon_{m}^{o}, \delta \varepsilon_{12}^{o}\right)^{T}$, i.e., displacement of the center of the flaw $\delta y_{1}^{o}, \delta y_{2}^{o}$, rotation of the flaw $\delta \omega^{o}$, isotropic dilatation


Figure 5. L-SHAPE FLAW: FREQUENCY $\omega_{3} / \omega_{n}=1.47$ (L-SHAPE INITIAL FLAW)


Figure 6. L-SHAPE FLAW: INITIAL, INTERMEDIATE AND FINAL SHAPE AND POSITION OF THE FLAW FOR FREQUENCY $\omega_{1} / \omega_{n}=$ 0.53 (CIRCULAR INITIAL FLAW; RESTRICTED GEOMETRICAL VARIATION)
$\delta \varepsilon_{m}^{o}$, and distortion $\delta \varepsilon_{12}^{o}$
Again $M=20$ experimental measures have been provided. The initial, intermediate and final position of the flaw are shown in Figures 9 to 10 for the frequencies $\omega_{1}$ and $\omega_{2}$ respectively. For this case it has not been possible to attain convergence for the highest frequency

## CONCLUSIONS

The derivation of the variation boundary integral equation $(\delta$ BIE) for the Helmholtz equation is presented in this paper. This $\delta$ BIE can be use to solve identification inverse problems for 2D acoustic fields, although its extension to more complicated problems is straightforward. The integral equation is one order of singularity higher that the primary BIE, but it can be easily handle. The continuity requirements for the variation of the geometry and for the gradient of the potential is ensured by the Multiple Collocation Approach. This approach is simple and reliable and easily implemented in a conventional Boundary Element code.


Figure 7. L-SHAPE FLAW: FREQUENCY $\omega_{2} / \omega_{n}=0.8$ (CIRCULAR INITIAL FLAW; RESTRICTED GEOMETRICAL VARIATION)


Figure 8. L-SHAPE FLAW: FREQUENCY $\omega_{3} / \omega_{n}=1.47$ (CIRCULAR INITIAL FLAW; RESTRICTED GEOMETRICAL VARIATION)


Figure 9. L-SHAPE FLAW: FREQUENCY $\omega_{1} / \omega_{n}=0.53$ (CIRCULAR INITIAL FLAW; FULL GEOMETRICAL VARIATION)

To parameterize the variation of the defect geometry, the assumed flaw is translated, rotated and strained, as if it were in a virtual strain field. A clear advantage of this approach is that the assumed flaw can have any shape (square, circle, ellipse, etc.) since what it is computed is its variation. Different flaw shape


Figure 10. L-SHAPE FLAW: FREQUENCY $\omega_{2} / \omega_{n}=0.8$ (CIRCULAR INITIAL FLAW; FULL GEOMETRICAL VARIATION)
can be assumed without difficulty, and without implementing a particular parameterization for each shape.

An iterative procedure is devised which allows to compute successive positions of the flaw, without minimizing any cost function. A series of examples demonstrated the effectiveness of the present approach. The procedure detects the position and approximate shape of the real flaw for most of the applications solved. The last example where the frequency is higher than the natural frequency of the problem and where the parametric variation of the geometry is more complex does not converge. We expect that a refinement in the solution of the system (39) will improve the convergence of the iterative algorithm.

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