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NEW RECONSTRUCTION ALGORITHMS IN ELECTRICAL IMPEDANCE TOMOGRAPHY BASED ON NONLINEAR OPTIMIZATION

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ABSTRACT

We consider the inverse conductivity problem in Electrical Impedance Tomography for which we focus on the case of important conductivity jumps. If we formulate the problem with a boundary element method we obtain a nonlinear system which can be solved by a Levenberg-Marquardt method. The important size of this system is reduced by simulating a global current injection. The resolution of the new nonlinear system allows us to reconstruct the first layer of the domain. Once this layer obtained we iterate the process to reconstruct the next layer by choosing an accurate starting point that is a conductivity distribution which minimizes the difference between measured and calculated potentials on the boundary. With this method we obtain a layer reconstruction algorithm which gives good results not only for small conductivity jumps but also for important jumps.

NOMENCLATURE

σ_i	Conductivity distributions
φ	Potential distribution
φ_m	Potential measurements
φ_h	Potential distribution for an homogeneous domain
$\delta\varphi$	Potential difference
ϕ	Potential distribution for a global injection
Ω_i	Domain
Γ_i	Boundary of a domain
Γ_{ij}	Boundary between 2 domains
E_M	Elementary solution of Laplace equation
δ_M	Dirac distribution at point M
ν_i	Outward unit normal

INTRODUCTION

Electrical Impedance Tomography (EIT) seeks to recover the interior electrical conductivity of an inhomogeneous conducting object by means of low–frequency voltage and current measurements at the boundary. The application of EIT is, for example, in medical and geophysical fields where low–resolution images are adequate, when alternative techniques do not exist, or when inexpensive systems are necessary.

The EIT is related to a nonlinear inverse problem. For simplicity, we assume that the studied domain Ω with boundary Γ is a disk. We denote the electrical conductivity by σ and potential by φ. The problem can be formulated as a boundary value problem governed by the elliptic equation:

$$\nabla \cdot (\sigma \nabla \varphi) = 0 \text{ in } \Omega \tag{1}$$

with boundary conditions which can be Dirichlet, Neumann or mixed conditions.

If the conductivity σ is known in Ω, the forward problem consists in calculating the potential φ. The inverse problem for its part consists in reconstructing the conductivity distribution σ in Ω from potential measurements φ_m on Γ.

Some results on the uniqueness of the solution can be found in the works of Sylvester, Uhlmann and Nachman (Nachman et al.,1988), (Nachman,1995), (Sylvester and Uhlmann,1986), (Sylvester and Uhlmann,1987). In 1988, Alessandrini (Alessandrini,1988) has explained the low resolution of EIT images proving a weak dependance between the data and the conductivity. More recently Allers, Dobson and Santosa (Allers and Santosa,1991),(Dobson and San-

tosa,1994) have given some stability results.
Several methods of current injections can be used. We used injections by two adjacent electrodes and potential measurements by adjacent pairs of electrodes as proposed by Barber and Brown (Barber and Brown,1983). We can find an other method in the work of Isaacson (Isaacson,1986) based on an optimal current density which improve the "distinguishability" between two conductivity distributions.
As for the existence of a solution, since the EIT problem is ill–posed and the measured data are not exact, there is no exact solution to the inverse problem of EIT. So we have to ask how we can stabilize the ill–posed problem and then find an approximate solution. The first attempts in this direction are to linearize the ill–posed problem (like in (Barber and Brown,1983),(Barber and Brown,1986)). These linearization methods are very attractive because of their mathematical simplicity and computational fastness but they have the defect of ignoring the nonlinearity of EIT which implies difficulties in the case of important conductivity jumps. There exists another category of methods which acknowledges the nonlinearity and ill–posedness of EIT and attempts to treat it without linearization. Most of these approaches reformulate the EIT problem as a nonlinear optimization one, which requires the solution of the direct problem at each step of the iterative procedure used to obtain the solution.
Our approach is to stay as near as possible from the nonlinearity of the problem to take into account not only small conductivity jumps but also important conductivity jumps.

FORMULATION BY BOUNDARY ELEMENT METHOD

Let's consider Ω a medium of boundary Γ_0 and conductivity σ_0 perturbated by conductivity anomalies σ_i on subdomains Ω_i of boundary Γ_i. We call Γ_{ij} the boundary separating two subdomains Ω_i and Ω_j (see figure 1). We suppose the conductivity constant on each subdomain which implies that the potential is harmonic on each Ω_i.
If we consider E_M, the elementary solution of Laplace equation defined by:

$$\Delta E_M = -\delta_M \ \ in \ \ I\!R^2 \tag{2}$$

where δ_M is the Dirac distribution at point M, the second Green formula gives:

$$\int_\Omega [E_M \Delta(\sigma\varphi) - \sigma\varphi\Delta E_M]d\Omega = \sum_i \int_{\Omega_i} [E_M\Delta(\sigma_i\varphi) - \sigma_i\varphi\Delta E_M]d\Omega_i \tag{3}$$

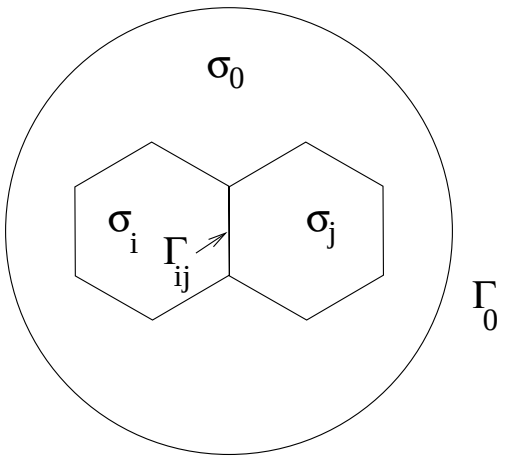

Figure 1. Hexagonal conductivity perturbations

that is:

$$\sigma(M)\varphi(M) = \sum_i \sigma_i \int_{\Gamma_i} [E_M \frac{\partial\varphi}{\partial\nu_i} - \varphi\frac{\partial E_M}{\partial\nu_i}]d\Gamma_i \tag{4}$$

where ν_i is the outward unit normal to Ω_i. If we take into account the continuity of the current normal component through the interface Γ_{ij}:

$$\sigma_i\frac{\partial\varphi}{\partial\nu_i} + \sigma_j\frac{\partial\varphi}{\partial\nu_j} = 0 \tag{5}$$

and that $\nu_i = -\nu_j$, we obtain the potential integral representation for $M \in \Omega$:

$$\sigma(M)\varphi(M) = \sigma_0 \int_{\Gamma_0} [E_M\frac{\partial\varphi}{\partial n_0} - \varphi\frac{\partial E_M}{\partial n_0}]d\Gamma_0 - \sum_{i,j}(\sigma_i - \sigma_j)\int_{\Gamma_{ij}} \varphi\frac{\partial E_M}{\partial\nu_i}d\Gamma_i \tag{6}$$

We show that we have the same formulation for $M \in \Gamma_0$ with:

$$\sigma(M) = \begin{cases} \sigma_i & \text{for } M \in \Omega_i \\ \sigma_0/2 & \text{for } M \in \Gamma_0 \\ (\sigma_i+\sigma_j)/2 & \text{for } M \in \Gamma_{ij} \end{cases} \tag{7}$$

If we call φ_h the potential distribution on the homogeneous domain of conductivity σ_0, equation (4) gives in this case,

for $M \in \Gamma_0$:

$$\frac{\varphi_h(M)}{2} = \int_{\Gamma_0} [E_M \frac{\partial \varphi}{\partial \nu_0} - \varphi \frac{\partial E_M}{\partial \nu_0}] d\Gamma_0 \qquad (8)$$

We can consider $\delta\varphi = \varphi - \varphi_h$ the difference between the perturbated and homogeneous potentials. Then, by difference between equation (6) and equation (8), we obtain, for a perturbated medium and for $M \in \Gamma_0$:

$$\frac{\sigma_0}{2}\delta\varphi(M) = -\sum_{i,j}(\sigma_i - \sigma_j) \int_{\Gamma_{ij}} \varphi \frac{\partial E_M}{\partial \nu_i} d\Gamma_i \qquad (9)$$

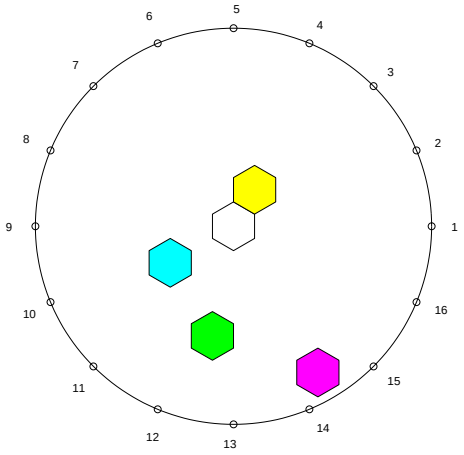

Figure 2. Five conductivity perturbations in an homogeneous domain with values (from the center to the boundary) 4, 10, 100, 300, 500 : *Real image*

If we write equation (6) and equation (9) for all the discretization points of $\overline{\Omega}$ and for each independent injection, we obtain a nonlinear system which unknowns are the conductivities σ_i and potentials φ (for each injection) inside the domain:

$$\begin{cases} \frac{\sigma_0}{2}\delta\varphi(M) & = -\sum_{i,j}(\sigma_i - \sigma_j) \int_{\Gamma_{ij}} \varphi \frac{\partial E_M}{\partial \nu} d\Gamma_{ij} \ \forall M \in \Gamma \\ \sigma(M)\varphi(M) & = \sigma_0 \int_{\Gamma} [E_M \frac{\partial \varphi}{\partial \nu} - \varphi \frac{\partial E_M}{\partial \nu}] d\Gamma \\ & \quad -\sum_{i,j}(\sigma_i - \sigma_j) \int_{\Gamma_{ij}} \varphi \frac{\partial E_M}{\partial \nu} d\Gamma_{ij} \ \forall M \in \Omega \end{cases} \qquad (10)$$

To solve such a system we use a Levenberg-Marquardt type algorithm: the subprogram SNLS1E of the numerical library CML (Common Mathematical Library) of SLATEC (Sandia, Los alamos, Air force weapons laboratory Technical Exchange Committee). We obtain good results with this algorithm as shown in figure 2 and 3. In this test, we put 5 perturbations of conductivity values (from the center to the boundary) 4, 10, 100, 300, 500, in an homogeneous medium of conductivity 1. The conductivity values are reconstructed with a precision of 10^{-7} but with important computation times. This is due to the size of the system: for example, for a small mesh of 19 hexagons, and only 16 electrodes we have a system of 2272 equations and 2035 unknowns. But the quality of the results obtained for important conductivity jumps encourage us to treat the problem in a nonlinear way. Now, the question is: how can we reduce the size ?

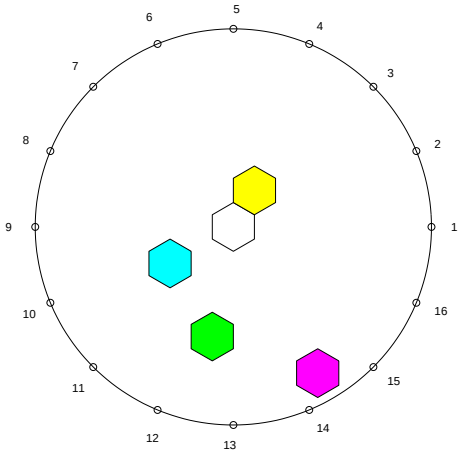

Figure 3. Image obtained by resolution of the nonlinear system.

GLOBAL INJECTION ALGORITHM

The important size of the system comes from the fact that we write the equations 10 not only for all the discretization points of $\overline{\Omega}$ but also for all independent injections.
If we take into account the linearity of this system with regard to the potential, we can group all the injections and then simulate one injection simultaneously by all electrodes (see figure 4) which comes to sum the corresponding equations. The system is then reduced to the size of a system corresponding to one injection . We can find a theoretical

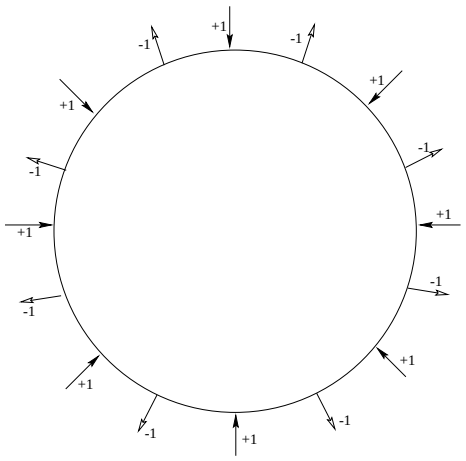

Figure 4. Global injection on the boundary

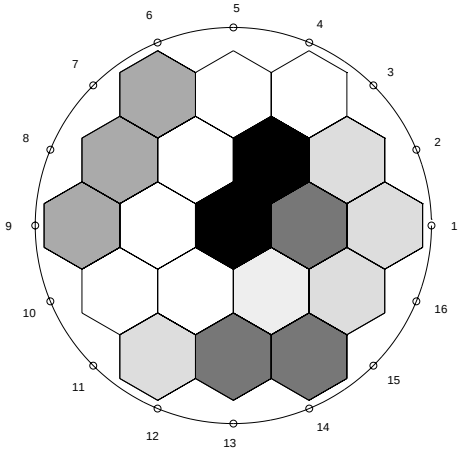

Figure 5. An example of perturbated domain.

justification of this in the work of Kohn and Vogelius (Kohn and Vogelius,1984).
If we call $\phi = \sum_{i=1,3,\ldots,N-1} \varphi_i$ where N is the number of injections and φ_i the potential corresponding to injection i, we can write the new system,for all the discretization points of $\overline{\Omega}$, as:

$$\begin{cases} \dfrac{\sigma_0}{2}\delta\phi(M) &= -\displaystyle\sum_{i,j}(\sigma_i-\sigma_j)\int_{\Gamma_{ij}}\phi\frac{\partial E_M}{\partial\nu}d\Gamma_{ij}\ \forall M\in\Gamma \\ \sigma(M)\phi(M) &= \sigma_0\displaystyle\int_\Gamma[E_M\frac{\partial\phi}{\partial\nu}-\phi\frac{\partial E_M}{\partial\nu}]d\Gamma \\ &\quad -\displaystyle\sum_{i,j}(\sigma_i-\sigma_j)\int_{\Gamma_{ij}}\phi\frac{\partial E_M}{\partial\nu}d\Gamma_{ij}\ \forall M\in\Omega \end{cases} \tag{11}$$

To solve this nonlinear system we use the same Levenberg-Marquardt type algorithm. If we test this method on a domain with small conductivity jumps (figure 5), with values between 0.9 and 1.5, we obtain a good quality image as shown in figure 6 with a precision of 10^{-4} for the conductivity values. Moreover, as expected, the computation time is decreased significantly.

If we look at different iterations of the algorithm, we can see a layer reconstruction begining with the layer near the boundary and making way to the center. In the case of important conductivity jumps (in the same configuration as in figure 5 but with conductivity values from 1 to 1000) we quickly reconstruct the first layer of the domain (as shown in figure 7) with a good precision but it's impossible to reconstruct the other layers. In fact, when we have reconstructed the first layer, the function, which must be minimized, takes very small values. So it becomes difficult to minimize it without entering in an instability domain. It comes from the fact that, grouping the injections, we loose the "distinguishability" power of separated injections. It's also well known that the first layer has a big influence on the measured potentials, hiding the other layers.
Once we have reconstructed the first layer, we have a new

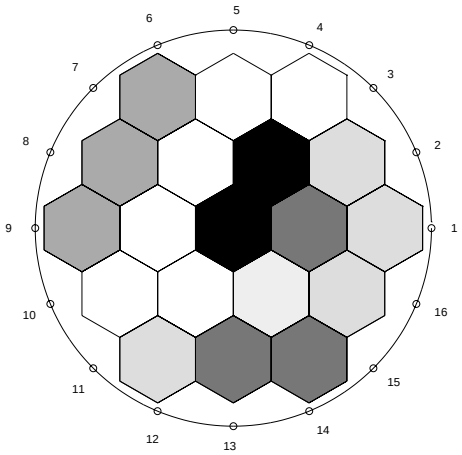

Figure 6. Image obtained by the Global Injection Algorithm on small conductivity jumps.

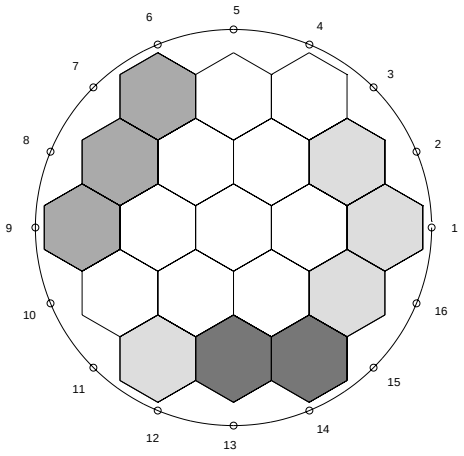

Figure 7. Image obtained by the Global Injection Algorithm for important conductivity jumps.

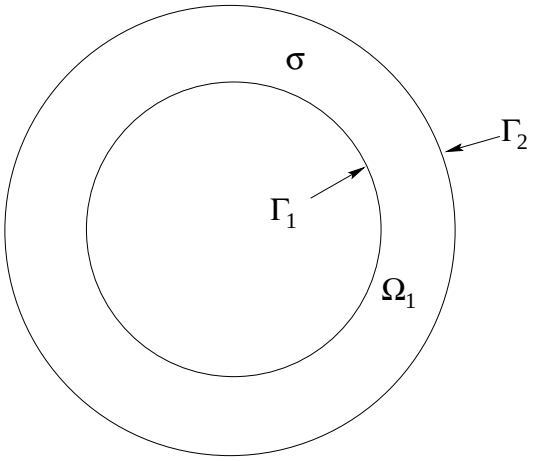

Figure 8. One layer of the domain

information on the domain. So the question is: how can we use this information to reconstruct the next layers ?
One way to answer to this question is to think to a "layer" algorithm. In fact we reconstruct the first layer from the knowledge of the potentials and currents on the exterior boundary (Γ_2 in figure 8) of the layer. Once the layer reconstructed we can try to calculate the potentials and currents on the interior boundary (Γ_1 in figure 8) to iterate the process for the next boundary. We have to solve a Cauchy problem which is ill-posed. To solve it we can use the Hibert Uniqueness Method proposed by JL Lions (Lions,1988) which is based on the exact controlability in ill-posed problems (we can find some results on this method in (Dai and Marsili,1993)). This method is very attractive in theory but in this case, due to the important sensibility of the problem, we are confronted to the classical problem of the derivatives approximation.
If we can't strip the layer, we must use it to iterate the process and reconstruct the other layers.

LAYER ALGORITHM

Once the first layer reconstructed, we can ask what initial conductivity we must take in the other layers to restart the process without falling in an instability domain. This is the difficulty for all optimization problems: the initial point value.
If we take an arbitrary conductivity value we fall in a high instability domain. Our first attempt was to take an initial conductivity value equal to a mean conductivity (with regard to the real conductivity values) which isn't utopian because we can calculate an approximation of such a conductivity by measuring the energy dissipation in the domain. We obtained a convergence of the Levenberg-Marquardt algorithm only in some simple cases.
Then, we tried to start as near as possible from the solution by taking an homogeneous conductivity in the center of the domain which minimize the difference on the domain boundary between calculated and measured potentials. To find this "optimal" conductivity we consider the unknown conductivities in the center of the domain as one unknown homogeneous conductivity (the conductivities on the first layer have already been calculated). When we solve the nonlinear system we find in just a few iterations (around 4 iterations) the "optimal" conductivity. In the case of figure 7 with conductivity values from 1 to 1000, the "optimal" conductivity value was about 2.23 which gives a relative error on the boundary potentials shown in figure 9. The peak is related to the electrod where the potential is near 0 and where the error is not significant. If we don't consider this peak we obtain a mean error of 2%. Moreover if we calculate the potentials inside the domain (with the first layer reconstructed and the "optimal" conductivity) and compare them to the real ones, we obtain the relative error shown in figure 10. We can see that the error is quite small in the right side of the figure which is related to the potentials in the first layer of the domain.
Following these results we can say that the "optimal" conductivity found minimizes not only the error on the potential on the boundary but also the error ont the potential in the first layer of the domain. With this new starting point,

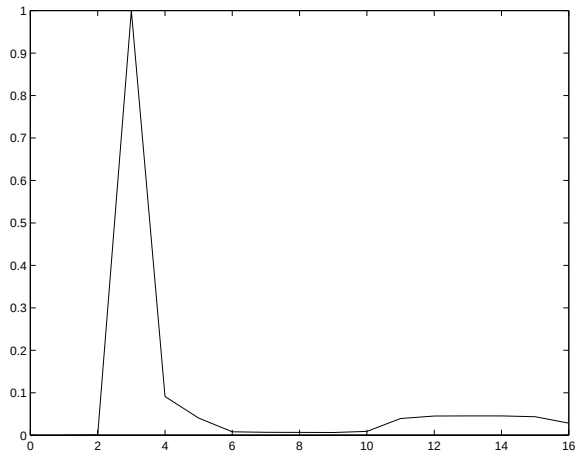

Figure 9. Relative error on measured potentials at the electrodes for the "optimal" conductivity

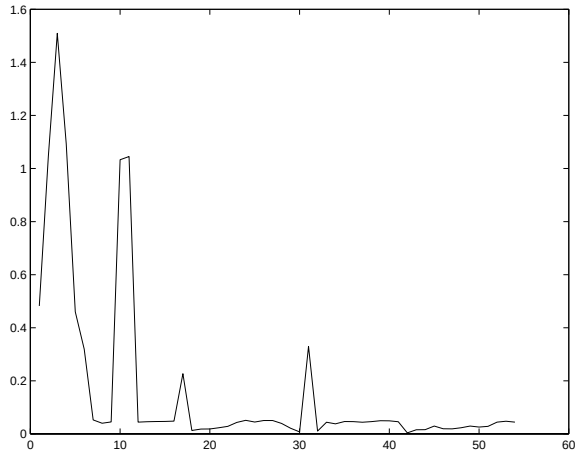

Figure 10. Relative error on potentials inside the domain for the "optimal" conductivity

we reconstruct the next layer with a good precision and iterating this by calculating a new "optimal" conductivity in the center of the domain at each step, we reconstruct all the domain with a relative error on conductivities between 0.1% (near the boundary) and 1.5% (at the center). These results were obtained in the same case as figure 6 but for important conductivity jumps (conductivity values from 1 to 1000).

So we have the following layer algorithm:

- **Initial step:** reconstruction of the conductivity on the first layer of the domain with the global injection algorithm,
- **Step 1:** Computation of an "optimal" conductivity by minimizing the difference between calculated and measured potentials on the boundary,
- **Step 2:** Computation of the corresponding potential distribution in the domain and on the boundary,
- **Step 3:** Reconstruction of the next layer, taking the conductivity and potentials calculated in step 1 and 2 as a new starting point for the global injection algorithm,
- **Iteration:** Go to Step 1 and so on until we have reconstructed all the domain.

CONCLUSION

Our aim was to work on domains with important conductivity jumps, a case where linearization techniques have some difficulties to work. With the formulation of the problem by a boundary element method, we obtain a nonlinear system where the unknowns are the conductivities and potentials inside the domain. This system can be solved by a Levenberg-Marquardt algorithm which gives good results but with a quite long computation time due to the size of the system. To reduce this size we consider one global injection on the boundary which allows us to reconstruct the first layer of the domain. If we calculate a good starting point we can iterate the algorithm and then reconstruct, layer after layer, all the domain with a good precision not only for small but also important conductivity jumps.
We used the same mesh for the direct and inverse problems which explain the important quality of the results. But if a method doesn't work on the same mesh it can't work in a real case. So the quality of the results obtained lead us to think that this algorithm could give quite good results on different meshes, which we are testing now. This is a difficult problem in inverse problems : how to adapt the mesh to the reality ? The problem of the injection type also can been studied testing our algorithm with, for example, an optimal current density.
It's obvious that, even with improvements, this nonlinear approach can't be used for real-time applications but it can be usefull for applications where time is less important than the results quality.

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