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REGULARISED FUNCTIONAL CALCULI AND THEIR APPLICATION TO THE NUMERICAL SOLUTION OF LINEAR INVERSE PROBLEMS

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ABSTRACT

Regularised functional calculi (RFC) for solving linear ill–posed problems are presented. Being very flexible with respect to the different types of *a priori* information about the problem to be solved, RFC provide the construction of the computationally effective regularising algorithms using only algebraic operations at ordinary numerical functions defined on the spectrum of a certain operator associated with the problem. Modelling techniques for obtaining regularised solutions via RFC are discussed and some numerical examples are presented.

INTRODUCTION

From a computational point of view, the key feature of regularising algorithms (RAs) is that they provide the stable inversion of ill–conditioned matrices. From a mathematical point of view, this inversion can be treated as the problem of the best approximation of an unbounded linear operator by bounded linear operators. S.B. Stechkin (Stechkin, 1967) was the first to formulate and solve rigorously this problem. The consequent developments were based mainly on the variational approach that prevailed in the theory of ill–posed problems for the last three decades. Although the general spectral method for constructing the regularising operators for linear ill–posed problems was proposed more than thirty years ago (Bakushinsky, 1967), it was not widely used in the scientific community. Perhaps this was mainly because of the equivalence of the α–approximating families of operators obtained from both the variational and spectral approaches. This equivalence takes place if *a priori* information about the problem to be solved is used in terms of the triple (A, y_δ, δ), where A is the given operator, y_δ denotes the measured data, and $\delta > 0$ is the level of errors in the data. If the other types of *a priori* information are used, then either this equivalence does not take place or using the variational approach meets serious difficulties. In this case, the spectral approach becomes the effective numerical technique for solving many linear ill–posed problems. In Refs. (Arsenin *et al.*, 1985), (Arsenin *et al.*, 1989) the concept of local regularisation was proposed on the basis of the spectral approach, and in Ref. (Tikhonov *et al.*, 1987) this concept was successfully applied to constructing the RAs for computer tomography.

The purpose of this note is to present regularised functional calculi (RFC) as the practically effective numerical technique for solving linear inverse problems of particular interest to engineering and remote sensing.

BACKGROUND MATERIAL

Let $\mathcal{H}$ be a complex separable Hilbert space, and A be a linear operator acting in $\mathcal{H}$. For brevity, let us assume that the operator A is selfadjoint and positive definite. Otherwise, one should consider either its polar decomposition $A = \mathcal{I}(A^*A)^{1/2}$, where $\mathcal{I}$ is a partly isometric operator acting from $\overline{\mathcal{D}(A^*)}$ onto $\overline{Im(A)}$, or the operator A^*A. Let us first assume that the operator A is bounded. In this case, it follows from the spectral theorem (Berezansky, 1996) that there exists a resolution of the identity $E_A(\lambda)$ on the σ–algebra of Borel subsets of the real axis, such that the oper-

ator function $f(A)$ can be represented as a spectral integral

$$f(A) = \int_0^{\|A\|} f(\lambda) dE_A(\lambda), \qquad (1)$$

where the function $f(\lambda)$ belongs to the algebra $\mathcal{A}$ of bounded functions whose domain is a certain segment containing the spectrum of the operator A. It is therefore natural to define a functional calculus for the operator A as follows.

The map $f(\lambda) \to f(A), f(\lambda) \in \mathcal{A}, f(A) \in \mathcal{L}(H)$, that transforms the identity function $f(\lambda) \equiv 1$ to the identity operator, and the function $f(\lambda) = \lambda$ to the operator A is said to be a functional calculus for A.

It is clear that the functional calculus determined by the spectral integral (1) generates a homomorphism of $\mathcal{A}$ to the algebra $\mathcal{L}(H)$ of bounded operators in the Hilbert space H. It is also convenient to consider the inverse problems in terms of an operator equation of the first kind

$$Ax = y, \qquad x, y \in H. \qquad (2)$$

For brevity, we assume that the map A is a one–to–one operator. In this case, there exists the inverse A^{-1} of the operator A, but it is, generally speaking, unbounded. One can, however, define the functional calculus for the operator A^{-1} by analogy with the definition indicated above. Specifically, let us consider a set of all functions $f(\lambda)$ that are measurable with respect to an algebra of subsets of the real axis and almost everywhere finite with respect to an operator–valued measure $E_A(\lambda)$, i.e., $\mu(\{\lambda \in R : |f(\lambda)| = \infty\}) = 0$. In this case, one can define the operator

$$f(A^{-1}) = \int_0^\infty f(\lambda) dE_A(\lambda), \qquad (3)$$
$$D(f(A^{-1})) = \{y \in H : \int_0^\infty |f(\lambda)|^2 d(E_A(\lambda)y, y) < \infty\}.$$

Since there is no guarantee that the measured data y_δ belong to the domain $D(f(A^{-1}))$, the functional calculus defined by (3) needs to be regularised. Let us consider a set M of all functions that are measurable with respect to $E_A(\lambda)$.

The δ–parametric function $f_\delta(\lambda) \in M$ is said to be the δ–approximating function if $\lim_{\delta \to 0} |f_\delta(\lambda)\lambda - 1| = 0$.

Let $\mathcal{F}_\delta$ be a collection of all δ–approximating functions. Then, one can define the regularised functional calculus for A^{-1} as follows.

The functional calculus for A^{-1}, defined by the spectral integral (3), is said to be RFC if $f(\lambda) \in \mathcal{F}_\delta$ and the deviation of $f(A^{-1})$ from the operator A^{-1}, i.e.,

$$\Delta(f, y, \delta) = \sup\{(f(A^{-1})y_\delta, A^{-1}y)^{1/2} : (y, y_\delta)^{1/2} \le \delta\},$$

converges to zero as $\delta \to 0$.

Let $f_\delta(\lambda)$ denotes an arbitrary function belonging to the set $\mathcal{F}_\delta$. It is clear that any regularised functional calculus $f_\delta(A^{-1})$ is the regularised operator for the problem (2). Thus, the RFC reduce the problem of construction of the regularising operators to the problem of finding the appropriate δ–approximation of a spectral function generated by the inverse operator A^{-1}.

It is convenient sometimes to consider an inverse problem in terms of calculating the values of a linear unbounded operator instead of the solution of Eq. (2). To this point, one can formulate two problems. First, given a linear unbounded operator $U = A^{-1}$ acting in H and a certain approximation $y_\delta \in H$ of an element $y \in H$, such that $(y_\delta, y)^{1/2} \le \delta$. Find the element $Uy_\delta = x_\delta$, such that $x_\delta \to Uy = x$ as $\delta \to 0$. It is obvious that if the operators A and U are invertible, then these problems are equivalent each other because of the equalities $U = A^{-1}$ and $A = U^{-1}$. In this case, one can construct the RFC for the operator U using the resolution of the identity of the operator A. Second, given a linear unbounded operator A and a certain approximation $x_\delta \in H$ of an element $x \in H$, such that $(x_\delta, x)^{1/2} \le \delta$. Find the element $Ax_\delta = y_\delta$, such that $y_\delta \to Ax = y$ as $\delta \to 0$. The reason why we consider the second problem is that the map A can be a differential operator. In this case, we should construct the RFC using the resolution of the identity of the unbounded operator. In the latter case, the definitions need to be changed. Specifically, the following conditions should be satisfied:

$$\lim_{\delta \to 0} |f(\lambda)\lambda - \lambda| = 0,$$
$$\Delta(f, x, \delta) = \sup\{(f(A)x_\delta, Ax)^{1/2} : (x, x_\delta)^{1/2} \le \delta\}.$$

The general scheme for computing the RFC can be described as follows.

1. Formalise *a priori* information about the problem to be solved in terms of the spectral properties of the operator A, the desired solution x, the measured data y_δ, and errors in the data. *A priori* information is used to derive the appropriate δ–approximating functions $f_\delta(\lambda)$ that generate RFC for the operator A^{-1}. Often it is sufficient to consider only all rational δ–approximating functions $f_\delta(\lambda)$ on the real semiaxis in order to derive these functions.
2. Determine the spectral measure $E_A(\lambda)$ of the operator A. Since in computational practice the operator A is considered in a finite–dimensional Hilbert space, $\dim H = n$, the spectral measure is determined as $\sum_{k=1}^n \mathcal{P}(\lambda_k)$, where $\mathcal{P}(\lambda_k)$ is an orthogonal projector onto a set of all eigenvectors of the operator A, i.e., a

set $\{\varphi \in H : A\varphi = \lambda_k \varphi\}$. In other words, we should solve the eigenvalue problem for a matrix generated by the operator A. The computationally effective methods for solving this problem are indicated in Refs. (Faddeev, 1963), (Golub, Van Loan, 1996). However, it is not necessary to solve the eigenvalue problem if the δ–approximating functions $f_\delta(\lambda)$ generate the iterative processes.

3. Calculate RFC as

$$f_\delta(A^{-1})y_\delta = \sum_{k=1}^{n} f_\delta(\lambda_k)(y_\delta, \varphi_k)\varphi_k.$$

Since many linear inverse problems can be reduced to solving either a system of linear equations whose matrices are ill–conditioned or an integral equation of convolution type of the first kind, we show below how RFC can be applied to the numerical solution of these problems.

REGULARISED LEAST SQUARES SOLUTIONS VIA RFC

Let us assume that the operator A in Eq. (2) is generated by a certain matrix belonging to the vector space $R^{m\times n}$. The corresponding linear least squares problem is

$$\min\{\|Ax - y\|^2 : A \in R^{m\times n}, x \in R^n, y \in R^m, m \geq n\}. \quad (4)$$

If the condition number $\mu = \|A\|\|A^{-1}\|$ of the matrix A is sufficiently large, the problem (4) is called ill-conditioned. For brevity, let us assume that $ker A = \emptyset$. The most popular numerical method for solving the problem (4) is based on the well–known factorisation $A = UDV^*$ (Gantmacher, 1959), where $U \in R^{m\times m}, V \in R^{n\times n}$ are the orthonormal matrices, and the matrix $D \in R^{m\times n}$ has diagonal form, $D = diag(d_1, d_2, ..., d_n)$, where $d_\jmath > 0$, $\jmath = 1, 2, ..., r$ are singular values of matrix A. This decomposition is known in the contemporary literature as the singular value decomposition (SVD) of the original matrix A. Note that the columns of matrix V are eigenvectors of matrix A^*A and columns of matrix U are eigenvectors of matrix AA^*. Therefore, SVD is closely related to the Schur decomposition of matrix $A^*A = V\Lambda V^*$, where $\Lambda = diag\{\lambda_1, \lambda_2, ..., \lambda_n\}$, and $\lambda_i = d_i^2, (i = 1, 2, ..., n)$ are eigenvalues of matrix A^*A. Since the problem (4) is equivalent to the normal equation $A^*Ax = A^*y$, it is sufficient to consider the Schur decomposition of matrix A^*A, and the rational δ–functions f_δ in the form

$$f_\delta(\lambda) = \frac{\lambda}{\lambda^2 + q_\delta(\lambda)}. \quad (5)$$

where $q_\delta(\lambda) > 0$ are certain functions measurable with respect to $E_A(\lambda)$.

α–parametric RFC

There are a number of RAs based on either the Tikhonov's scheme of regularisation (Tikhonov, 1965), (Varah, 1973), (Elden, 1977) or the truncated singular value decomposition (TSVD) (Hansen, 1987). These RAs can, however, be generated by RFC as well. Indeed, if we choose the function q_δ in the form $q_\delta = \alpha(\delta)$, where $\alpha(\delta)$ is the regularisation parameter, then the functions $f_\delta(A^{-1})$ generate the well–known Tikhonov's regularisators. If we choose the functions $q_\delta(\lambda) > 0$ in the form

$$q_\delta(\lambda) = \begin{cases} 0 & \text{if } \lambda > \alpha(\delta) \\ \infty & \text{if } \lambda \leq \alpha(\delta), \end{cases} \quad (6)$$

then these functions generate the δ–approximations $f_\delta(\lambda)$

$$f_\delta(\lambda) = \begin{cases} \lambda^{-1} & \text{if } \lambda > \alpha(\delta) \\ 0 & \text{if } \lambda \leq \alpha(\delta), \end{cases}$$

that are equivalent the regularised TSVD. Note that we used above *a priori* information about the problem to be solved in terms of the triple (A, y_δ, δ). The α–parametric RFC allow the smoothed approximations of the desired solution because of the global level δ of errors in the measured data. Often we wish to reconstruct the "fine" structure of the desired solution. In this case, more *a priori* information is needed in order to provide this reconstruction. For example, such an information can be given in the form of the componentwise errors $|y_{\delta k} - y_k| \leq \delta_k, (k = 1, 2, ..., m)$. In this case, it is not feasible to construct α–aproximations because of the numerical regularisation parameter α. Fortunately, it is possible to construct RFC using the general scheme indicated above.

q_δ–parametric RFC

As an example, we consider the case of the spectral componentwise errors, i.e., $|\hat{b}_{\delta\jmath} - \hat{b}_\jmath| \leq \delta_\jmath$, where $\delta^2 = \sum_{\jmath=1}^{n} \delta_\jmath^2$, $\hat{b}$ and $\hat{b}_\delta$ are spectral decompositions of vectors $b = A^*y$ and $b_\delta = A^*y_\delta$, respectively. Here A^* is the transpose of the matrix A. The general scheme is, however, applicable for the other cases as well. In order to clarify why we need to construct the q_δ–parametric RFC, we show that there exists the q_δ–function that minimises the upper bound of the error $\|x_q - x\|^2$. Let $\hat{x}$ be the spectral decomposition of the vector x, i.e., $\hat{x} = V^*x$. Then, minimising the upper bound

of the error

$$\|x_q - x\|^2 \le \sum_{j=1}^{n} \frac{\lambda_j^2 \delta_j^2 + q_\delta^2(\lambda_j)|\hat{x}_j|^2}{[\lambda_j^2 + q_\delta(\lambda_j)]^2},$$

we obtain

$$\tilde{q_\delta} = \begin{cases} \infty & \text{if } \hat{x}_j = 0 \\ \frac{\delta_j^2}{|\hat{x}_j|^2} & \text{if } \hat{x}_j \neq 0. \end{cases} \tag{7}$$

Since the "optimal" function $\tilde{q_\delta}$ depends on the unknown solution x, it cannot be implemented in practice. One can, however, try to construct the other δ–functions which would be close in some sense to the "optimal" function. One class of such functions was described in Ref. (Arsenin *et al.*, 1985). According to the general scheme, one can consider a collection of vectors $q_\delta^{(k)} = \{q_{\delta 1}^{(k)}, ..., q_{\delta n}^{(k)}\}$ in the form

$$q_{\delta j}^{(k)} = \begin{cases} \delta_j^2 |\hat{x}_{\delta j}^{(k)}|^{-2} & \text{if } \hat{x}_{\delta j}^{(k)} \neq 0 \\ \infty & \text{if } \hat{x}_{\delta j}^{(k)} = 0, \end{cases} \tag{8}$$

so that the kth vector of the regularised spectral decomposition has the form

$$\hat{x}_\delta^{(k)} = \frac{\lambda}{\lambda^2 + q_\delta^{(k-1)}} \hat{b}_\delta, (k = 1, 2, ...). \tag{9}$$

It is easy to see that the vectors $q_\delta^{(k)}$ generate RFC for the pseudo–inverse of A if the initial vector $q_\delta^{(0)}$ generates such RFC, i.e., if the element $x_\delta^{(0)} = V\hat{x}_\delta^{(0)}$ converges to the desired solution as $\delta \to 0$. The initial approximation can be determined as $x_\delta^{(0)} = V \Lambda_{q^{(0)}} V^* b_\delta$, where

$$q_{\delta j}^{(0)} = \begin{cases} \infty & \text{if } |\hat{b}_{\delta j}| \le \delta_j \\ \lambda_j^2 \delta_j (|\hat{b}_{\delta j}| - \delta_j)^{-1} & \text{if } |\hat{b}_{\delta j}| > \delta_j. \end{cases}$$

and

$$\Lambda_{q^{(0)}} = diag\{\frac{\lambda_1}{\lambda_1^2 + q_1^{(0)}}, ..., \frac{\lambda_n}{\lambda_n^2 + q_n^{(0)}}\}.$$

Note that the initial approximation satisfies the variational problem

$$argmin\{\|x\|^2 : x \in R^n, |\lambda_j \hat{x}_j - \hat{b}_{\delta j}| \le \delta_j\}, \tag{10}$$

that generalises the well–known discrepancy method for solving ill–posed problems. Furthermore, there exists the converging limit function that has the form

$$\hat{x}_{\delta j}^{lim} = \begin{cases} 0 & \text{if } |\hat{b}_{\delta j}| < 2\delta_j \\ 0 & \text{if } |\hat{b}_{\delta j}| \ge 2\delta_j, |\hat{x}_{\delta j}^{(0)}| < B_j \\ \lambda_j^{-1} \hat{b}_{\delta j} C_j & \text{if } |\hat{b}_{\delta j}| \ge 2\delta_j, |\hat{x}_{\delta j}^{(0)}| = B_j \\ \lambda_j^{-1} \hat{b}_{\delta j} D_j & \text{if } |\hat{b}_{\delta j}| \ge 2\delta_j, |\hat{x}_{\delta j}^{(0)}| > B_j, \end{cases} \tag{11}$$

where $A_j = |\hat{b}_{\delta j}| \sqrt{|\hat{b}_{\delta j}^2 - 4\delta_j^2}$, $B_j = 0.5\lambda_j^{-2}(|\hat{b}_{\delta j}|^2 - 2\delta_j^2 - A_j)$, $C_j = 1 - 2\delta_j^2(|\hat{b}_{\delta j}|^2 - A_j)^{-1}$, $D_j = 1 - 2\delta_j^2(|\hat{b}_{\delta j}|^2 + A_j)^{-1}$. Analogously, the other RFC can be constructed for any other *a priori* information about the problem to be solved.

A Model Problem

To evaluate RFC indicated above, the computational experiment has been conducted. The test problem was initially formulated for the compact integral operator $A = \int_a^b K(t - t')dt', t \in [c, d]$ with the kernel function $K(t) = (1 + t^2)^{-1}$. After its discretisation the eigenvalues of the matrix generated by a finite–dimensional approximation of the operator A decreased rapidly. It allowed for modelling an ill–conditioned system of linear equations. The exact solution was defined as the unit vector $x = [1, 1, ..., 1]^T$. The typical values of m and n were equal to 1000 and 100, respectively. The exact right-hand side was perturbed by uniformly distributed random vector modelled noise. Each noise sample has been rescaled from 10% down to 0.01% of the norm of the right-hand side as follows

$$\tilde{y} = y + \delta \xi \|y\| \|\xi\|^{-1}$$

where $\delta > 0$ is the root mean square of the right-hand side. The quality of RFC has been characterized by the relative error of the regularised solution, i.e., $\varepsilon = \|x_q - x\| \|x\|^{-1}$. Calculations were carried out with double precision. Table 1 summarizes values of ε for the Tikhonov's scheme of regularisation using QR decomposition of the matrix A^*A, SVD and TSVD of the original matrix A, and the initial approximation of the RFC. Comparing the values of ε, one can notice that the RFC provide more accurate regularised solutions for any reasonable values δ. In our calculations we used the standard FORTRAN routines JACOBI and DSVD[1]. It was expected that the different results would appear when another routines were used. Table 2 contains

[1]These routines were implemented by Applied Mathematics Division of Argonne National Laboratory.

δ	QR	SVD	TSVD	RFC
10^{-8}	.581d-2	.595d-2	.385d-2	.202d-2
10^{-6}	.237d-1	.237d-1	.140d-1	.116d-2
10^{-4}	.570d-1	.571d-1	.475d-1	.310d-1
10^{-2}	.724d-1	.735d-1	.724d-1	.400d-1
$5 \cdot 10^{-2}$	.982d-1	.958d-1	.115d+0	.554d-1
10^{-1}	.138d+0	.133d+0	.213d+0	.664d-1

Table 1. Comparison of the relative errors for RA's based on the QR decomposition, SVD, TSVD and the RFC (initial approximation).

δ	QR	SVD	TSVD	RFC
10^{-8}	.151d-1	.169d-1	.835d-2	.613d-2
10^{-6}	.413d-1	.421d-1	.346d-1	.238d-2
10^{-4}	.482d-1	.477d-1	.474d-1	.320d-1
10^{-2}	.524d-1	.528d-1	.511d-1	.334d-1
$5 \cdot 10^{-2}$	.832d-1	.817d-1	.982d-1	.467d-1
10^{-1}	.128d+0	.123d+0	.204d+0	.575d-1

Table 2. Comparison of the relative errors for RA's based on the QR decomposition, SVD, TSVD and RFC (initial approximation).

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the results obtained by IDL standard routines[2]. Although the slightly different results were obtained, the difference was not essential. This means that both the α- and q_δ–parametric RFC are computationally stable with respect to the errors caused by the different approximate algorithms for computing eigenvalues and eigenvectors.

Regularised Mode Filtering

As an alternative to known matched–field processing (MFP), matched–mode processing (MMP) has been developed recently (Baggeroer, et al., 1993) in ocean acoustics for detection and localisation of a source in shallow water. The main advantage of MMP is its potentially low sensitivity to environmental noise. Since the effectiveness of MMP depends strongly on the accuracy of mode filtering, the mode filtering algorithms are the core of MMP. In order to outline the basic problem of mode filtering, let us consider an acoustic field generated by a narrow–band point source. Without loss of generality, let us assume that the oceanic waveguide is cylindrically symmetric, and the source is situated at the point (r_s, z_s). Then the complex pressure field received by a vertical hydrophone array situated in the far–field zone can be expressed as a finite series whose elements are the weighted propagating normal modes

$$p(z; r_s, z_s) = b \sum_{j=1}^{n} u_j(z) u_j(z_s) \frac{\exp(\imath k_{rj} r_s - \beta_j r_s)}{\sqrt{k_{rj} r_s}}, \qquad (12)$$

where $u_j(z)$, β_j, and k_{rj} are the modal eigenfunctions, the attenuation and the horizontal wavenumbers, respectively. The modal eigenfunctions satisfy the Sturm-Liouville problem, and they are orthogonal. The time dependence term will be neglected here because it is not essential for the arguments below. For similar reasons, we do also not explicitly define the complex–valued coefficient b. Defining the modal amplitudes as $x_j(r_s, z_s) = b \cdot u_j(z_s) \frac{\exp(\imath k_{rj} r_s - \beta_j r_s)}{\sqrt{k_{rj} r_s}}$, we represent Eq. (12) as a system of linear equations

$$Ax \equiv \sum_{j=1}^{n} a_{ij} x_j = p(z_i; r_s, z_s), (i = 1, 2, ..., m), \qquad (13)$$

where the matrix A acts from an n-dimensional vector space into an m-dimensional vector space. The case of a horizontal hydrophone array can be considered analogously. If the vertical hydrophone array is spanning the entire waveguide, and the number m is sufficiently large, i.e., good sampling of the mode shapes is performed, then the matrix A is expected to be quasi–unitary, so that $A^{-1} \cong A^*$, where A^* is a Hermitian matrix. However, real vertical hydrophone arrays do not span the entire waveguide, and they sample poorly the mode shapes. As a result, the matrix A is no longer quasi–unitary, and it is, generally speaking, ill–conditioned. To demonstrate the effectiveness of RFC, we simulated shallow water mode filtering. Specifically, in our simulation the Pekeris model of waveguide was used, in which the water channel depth was 80 m, and source depth and range were 30 m and 1000 m, respectively. The simulation used 50 Hz as the propagation frequency. We used the following noise model: $A^*Ax = A^*p + \xi$, $\xi = A^*N$, where the noise vector N was modelled as a complex Gaussian random vector, so that the random vector was also a complex Gaussian vector with zero mean and correlation matrix R_ξ. Since $\delta^2 \equiv \mathbf{E}\|\xi\|^2 = trR_\xi$, it was natural to use $diagR_\xi$ for estimating the spectral error mentioned above. The noise level was expressed in terms of SNR, i.e., $SNR = 10 \log \frac{\|A^*p\|^2}{\mathbf{E}\|\xi\|^2}$. Figures 1 and 2 plot the performance of the conventional pseudo–inverse algorithm, the Tikhonov's scheme of regularisation and RFC(initial ap-

[2]Interactive Data Language, a data analysis software product from Research Systems, Inc.

proaximation) for vertical arrays whose apertures span 0.75 and 0.5 of the channel depth, respectively.

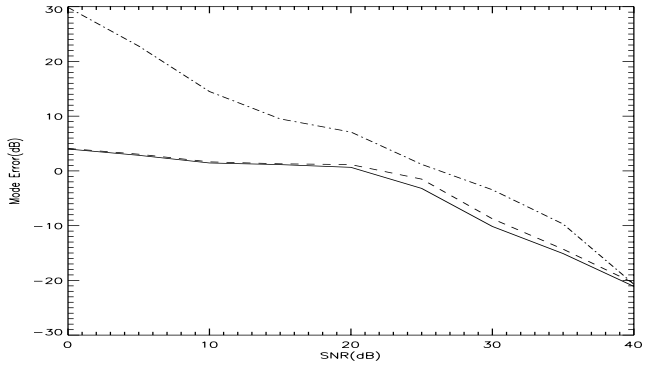

Figure 1. Comparison of total squared errors: the pseudo–inverse (dash–dot), the Tikhonov's scheme of regularisation (dotted) and RFC (solid).

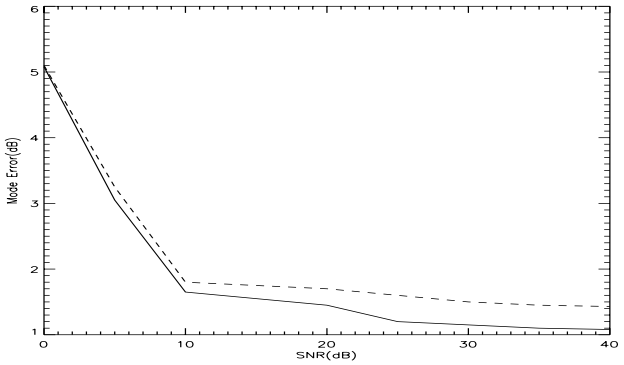

Figure 2. Comparison of total squared errors: the Tikhonov's scheme of regularisation (dotted) and RFC (solid).

As expected, the pseudo–inverse algorithm does not work well even for the 20– and 40-dB SNR cases for both good (Figure 1) and poor (Figure 2) spanning. The filters based on the Tikhonov's scheme of regularisation and RFC provide reasonably good performance for the interval 20–40 dB SNR when the array spans more than 0.75 of the water column. However, the RFC filter provides the slightly better results. This is qualitatively true for the poorly spanned water column, but the mode error increases dramatically for all SNR cases and mode filters.

THE INTEGRAL EQUATION METHOD

Many boundary value problems can be reduced to the integral equations using the Green's theorems. This feature allows the computationally effective formulations of many complex applied problems that are of particular interest to engineering. As an example, we indicate below inverse scattering. Specifically, we show how RFC can be exploited within the framework of the integral equation method.

Integral Equations of Convolution Type of the First Kind

Let us consider an integral equation of convolution type of the first kind

$$Ax \equiv \int_{-\infty}^{\infty} K(t-\tau)x(\tau)d\tau = y(t), -\infty < t < \infty \qquad (14)$$

In this case, the spectral measure $E_A(\lambda)$ can be calculate via the Fourier transform. It is therefore sufficient to define the δ–approximating functions $f_\delta(\lambda)$ in the frequency domain. In particular, we can consider the rational δ–approximating functions in the form $f_\delta = K(-\lambda)[|K(\lambda)|^2 + q_\delta]^{-1}$, where $K(\lambda)$ is the Fourier transform of the kernel $K(t)$ of Eq. (14). The spectral error can be incorporated as $|y_\delta(\lambda) - y(\lambda)| \leq \sigma(\lambda)$, where $\int_{-\infty}^{\infty} \sigma^2(\lambda)d\lambda = \delta^2$. After this, the specific functions q_δ can be derived as before. Figure 3 shows the results of numerical experiment demonstrating the computational effectiveness of RFC(initial approximation).

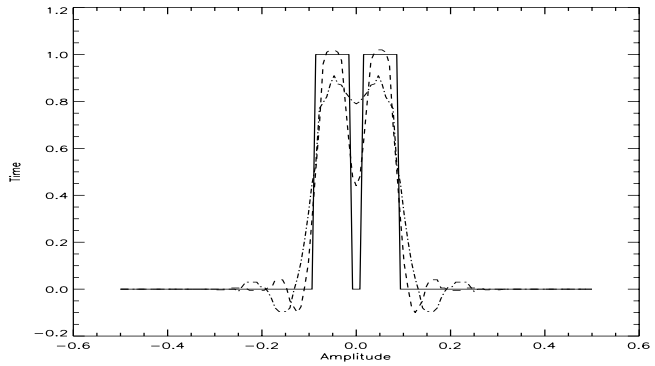

Figure 3. Comparison of numerical experiment: the exact solution(solid), the Tikhonov's scheme of regularisation(dash–dot) and RFC(dotted). The kernel has the form $K(t) = \exp[-100(t-0.5)^2]$. The level of errors corresponds to 3% of the maximum of the exact right–hand side of Eq. (14).

Inverse Scattering In Ocean Acoustics

The scattering by obstacles submerged in an oceanic waveguide or buried in the near–seafloor sediments and by

the volume inhomogeneities is usually considered separately. Meanwhile, the real ocean contains the volume inhomogeneities, such as bubble clouds, rough interfaces, internal waves, etc. Although the specific mathematical model of an ocean waveguide depends on the specific applied problem, we consider, for brevity, the three layer model that can be of particular interest to acoustic remote sensing. The upper and lower layers of this model are the homogeneous air and bottom background. The middle layer represents the arbitrarily inhomogeneous seawater and the near–seafloor sediments that may include several obstacles. The mathematical model can be represented in terms of the Cauchy problem for the Chernov's wave equation with the variable mass density $\rho(\mathbf{r})$ and sound speed $c(\mathbf{r})$

$$\rho(\mathbf{r})div(\rho^{-1}(\mathbf{r})gradp(\mathbf{r},t)) - c^{-2}(\mathbf{r})\frac{\partial^2 p}{\partial t^2}(\mathbf{r},t) = -f(\mathbf{r},t) \quad (15)$$

$$p(\mathbf{r},t) = 0, \frac{\partial p}{\partial t}(\mathbf{r},t) = 0, t < 0. \quad (16)$$

We assume that the conjugation conditions are satisfied at the air–water and water–sediments interfaces, and the Sommerfeld radiation condition satisfies at infinity. We also assume that there is a bounded region $\mathcal{D}$ containing a rigid body, so that the Neumann condition is satisfied on its boundary $\partial\mathcal{D}$. Although it is impossible to obtain the analytical solution of the Cauchy problem, this problem can be reduced to the integral equation using the Green's theorems. Indeed, applying the Fourier transform to Eqs. (15)–(16), we arrive to the boundary value problem for the Helmholtz equation for every fixed frequency ω. Introducing then a substitution $u(\mathbf{r},\omega) = p(\mathbf{r},\omega)/\sqrt{\rho}(\mathbf{r})$ and applying the Green's theorems to the Helmholtz equation, we obtain the integral representation of the boundary value problem

$$u(\mathbf{r},\omega) = \int_{\mathcal{F}} \varphi(\mathbf{r},\omega)G(\mathbf{r},\mathbf{r}',\omega)d\mathbf{r} + \int_V U(\mathbf{r},\omega)G(\mathbf{r},\mathbf{r}',\omega)u(\mathbf{r}',\omega)d\mathbf{r}' + \int_{\partial\mathcal{D}} u(\mathbf{r}',\omega)\frac{\partial G}{\partial n}(\mathbf{r},\mathbf{r}',\omega)ds \quad (17)$$

where $V \subset R^3\backslash\overline{\mathcal{D}}$, $\varphi(\mathbf{r},\omega) = f(\mathbf{r},\omega)/\sqrt{\rho}(\mathbf{r})$, and $U(\mathbf{r},\omega)$ is the potential that is the known function of the mass density and sound speed. Note that the Green's function $G(\mathbf{r},\mathbf{r}',\omega)$ satisfies the equation $(\nabla^2 + \omega^2 c^{-2}(z))G = -\delta(\mathbf{r}-\mathbf{r}')$, the conjugation conditions at the air–seawater and seawater–bottom interfaces, and the radiation condition at infinity. The Green's function can be represented via the Sommerfeld–Weil integral. It is easy to see that the integral terms in the right–hand side of Eq. (17) represent the incident field, the scattered field from the inhomogeneity, and the scattered field from the rigid body. The impedance or Kirchhoff conditions can also be considered.

The integral representation (17) is the basis for the construction of mathematical models for both forward and inverse 3–D modelling. However, the properties of the integral operators generated by this representation depend essentially on the position of the vector $\mathbf{r}$. As a result, in the case of a forward problem, we obtain the system of linear integral equations. However, any inverse problem becomes nonlinear because of the unknown coordinates of a source, the shape of a scatterer, etc., and the acoustic field $u(\mathbf{r},\omega)$ inside the inhomogeneity and on the surface of the reflector. In this case, after discretisation of this system we arrive to a system of nonlinear equations with respect to the unknowns indicated above. This system can be approximatively solved by the iteratively regularised Newton's methods if there is the Frěchet derivative of the operator generated by the system of integral equations.

In certain cases, it is possible to linearise an inverse problem. As an example, we consider one scheme of acoustic sounding based on synthetic aperture sonar technique. For brevity, let us assume that a scatterer is situated in a homogeneous waveguide with the constant mass density and sound speed c. We assume, however, that the surface of the scatterer is diffusive. In this case, one can formally define the apparent reflection coefficient $\kappa(\mathbf{r})$ on the surface of the diffusive scatterer and consider the generalised Kirchhoff conditions in the form

$$u_{sc}(\mathbf{r},\mathbf{R};\omega) = \kappa(\mathbf{r})\chi(\mathbf{r},\mathbf{R})u_i(\mathbf{r},\omega)$$
$$\frac{\partial u_{sc}}{\partial n}(\mathbf{r},\mathbf{R};\omega) = -\kappa(\mathbf{r})\chi(\mathbf{r},\mathbf{R})\frac{\partial u_i}{\partial n}$$

where $\mathbf{R}$ is the radius vector connecting the sonar and an arbitrary point $\mathbf{r}$ belonging to the surface of the scatterer, $\chi(\mathbf{r},\mathbf{R})$ is the characteristic function of the exposed area of the surface $\partial\mathcal{D}$. The physical sense of the apparent reflection coefficient $\kappa(\mathbf{r})$ is that it is the reflection coefficient of a plane wave from the elementary plane tangential to the diffusive surface of the scatterer at the point $\mathbf{r} \in \partial\mathcal{D}$. In many cases of particular interest to remote sensing, it is natural to formulate the inverse problem in terms of recovering the function $\kappa(\mathbf{r})$ from measurements of the scattered field in the far–field zone. Then, using the generalised Kirchhoff conditions, one can obtain from Eq. (17) the linear integral equation of the first kind with respect to $\kappa(\mathbf{r})$. This integral equation can be written as follows

$$u_{sc}(\mathbf{R},\omega) = \frac{i\omega}{(4\pi)^2 c}\int_0^\infty \exp(2ir/c)r^{-2} \cdot \int_{\mathcal{L}_r(\mathbf{R})} P(\alpha,\beta;\omega)\kappa(\mathbf{r})\chi(\mathbf{r},\mathbf{R})\sin 2(\mathbf{R}-\mathbf{r}\hat{}\,\mathbf{n})dl(\mathbf{r}) \quad (18)$$

where $\mathcal{L}_r(\mathbf{R}) = \{\mathbf{r} : \mathbf{r} \in \partial\mathcal{D}, |\mathbf{R}-\mathbf{r}| = r\}$ is a spatial curve on the surface $\partial\mathcal{D}$, and $P(\alpha,\beta;\omega)$ is the far–field pat-

tern that is determined from the asymptotic form of the solution $u(\mathbf{r},\omega) = \frac{\exp(\imath\omega r/c)}{r} P(\alpha,\beta;\omega) + O(r^{-2}), r \to \infty$. The curve $\mathcal{L}$ is a section of the surface $\partial\mathcal{D}$ by the front of the incident wave that moves along the vector $\mathbf{R}$. It is clear that Eq. (18) is the typical equation of integral geometry allowing the tomographic interpretation of the synthetic aperture sonar technique. In particular, if the scatterer is quasi–plane, then this equation can be transformed to one of the basic equations of linear computer tomography, e.g., the inverse Radon transform, that can be approximately solved via RFC (Tikhonov *et al.*, 1987). As an example, we indicate Figure 4 illustrating the effectiveness of this approach.

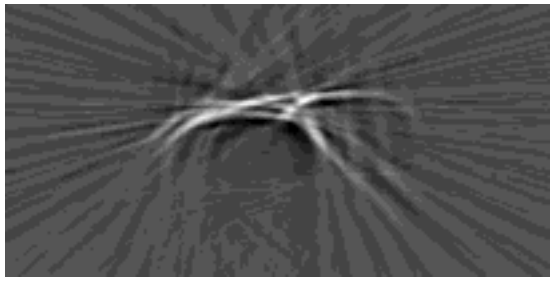

Figure 4. Acoustic image of a skin diver obtained from the data of a real–life acoustic experiment using the Kirchhoff approximation (Timonov, 1990).

CONCLUSION

This paper presents RFC as the practical numerical technique for solving linear inverse problems of particular interest to engineering. The RFC are shown to be a generalisation of the spectral interpretation of the Tikhonov's scheme of regularisation. Roughly speaking, the RFC provide the spectral local regularisation of the problem, i.e., each spectral component of the solution is filtrated by the corresponding value of the variable regularisation parameter. This form of regularisation is well–suited to the different types of *a priori* information about the problem to be solved. It is demonstrated that the RFC provide the computationally effective solutions to the least squares problem as well as to the integral equations of the first kind.

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