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AN ALGORITHM CONSTRUCTION METHOD FOR INVERSE PROBLEM SOLUTION STABLE TO THE DESCRIPTION ERRORS

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ABSTRACT

In IHCP solving one has very often to use mathematical models approximately describing the real processes. On the one hand, it may be caused by absence of just an exact model; on the other hand, by conscious employing an insufficiently adequate model, however, more suitable from problem solution viewpoint. As an example, a problem of determining a heat flux applied to the structure member surface made of a material, the thermal properties of which greatly depend on temperature can be considered. A linear model of the process in this case is not quite adequate, however, it allows to construct fast-acting algorithms. But the change-over to a linear model is justified only in the case when solution errors caused by this change-over have an acceptable value.

The paper proposes an algorithm construction method stable to description errors. The essense is in the following: a description error is conventionally represented in the form of expansion by a certain system of basic functions, the first expansion components included into a set of unknown parameters. In addition, some a priori restrictions are imposed on these components, which results in the extended problem statement of Tikhonov two-parameter functional optimum control. It has been shown that this approach is more effective in comparison with a generalized residual principle and some relative methods. The results of numerical simulation are presented.

INTRODUCTION

In solving inverse heat conduction problems (IHCP) one has very often to use mathematical models approximately describing real processes. It may be caused either by absence of an exact model or a conscious change-over to a model insufficiently adequate but, however, more suitable in a certain sense. The paper considers a problem of determining a heat flux applied to the flat plate surface possessing nonlinear thermal properties. From the viewpoint of computational expenditure it is convenient to replace a non-linear model by a linear one but there arise some mathematical description errors at that. An algorithm construction method stable to description errors is given below. The essense of this method is in the following; a description error is conventionally represented in the form of expansion by a certain system of basic functions, the first expansion components included into a set of unknown parameters. In addition, some a priori restrictions are imposed on these components, which results in the extended problem statement of optimum control for the Tikhonov two-parameter functional.

NOMENCLATURE

τ - time
x - coordinate
b - plate thickness
d_t - time interval
$T(\tau, x)$ - temperature
$C(T)$ - volumetric heat capacity
$\lambda(T)$ - thermal conductivity
k - sensor number
x_k - temperature sensor location coordinates
$f_k^*(\tau)$ - temperature measurements
$\xi_k(\tau)$ - sensor measurement error.

PROBLEM STATEMENT AND SOLUTION

Consider a problem of IHCP useful for practical applications that can be referred to the boundary-retrospective statements according to classification (Alifanov, 1994). The problem is in recovering the body temperature field in a particular space-and-time domain by the results of temperature measurements at finite number of points inside the body.

Consider the following IHCP statement for infinite plate at time interval :

$$C(T)T_\tau - \left(\lambda(T)T_x\right)_x = 0, \quad x \in [0,b], \quad \tau \in [0, d_t), \tag{1}$$

$$T(0,x) = T_0(x), \qquad T(\tau,0) = T_1(\tau), \qquad \lambda\left(T(\tau,b)\right)T_x(\tau,b) = 0,$$

$$x \in (0,b), \qquad\qquad \tau \in (0,d_t),$$

(2) $\quad T(\tau,x_k) + \xi_k(\tau) = f_k^*(\tau), \quad k = \overline{1,N}, \quad \tau \in (0,d_t),$

(3) It is necessary to determine functions $T_0(x)$ and $T_1(x)$ using the measurement data $f_k^*(\tau)$.

One of the most universal methods of nonlinear IHCP solution is considered in (Alifanov, 1994; Alifanov *et al.* 1988). The method is based on gradient algorithms of non-linear IHCP extremum statement optimization together with the iterational regularization principle. Each iteration should include both the numerical temperature field calculation and the adjoint problems solution to determine a goal functional gradient. Consider an approximate method of nonlinear IHCP Solution (1)-(3) allowing to avoid an iteration procedure and construct a temperature field observation algorithm in real or close to real time (Artyukhin & Gejadze, 1998; Alifanov *et al.* 1999).

As a result of some transformations and change-over to non-dimensional variables the initial problem (1)-(3) can be given (with previous notations remained) in the form

$$R_\tau - R_{xx} = -\left(\rho\widetilde{C}(R) - 1\right)R_\tau, \quad x \in [0,1], \quad \tau \in [0, d_t) \tag{4}$$

$$R(0,x) = R_0(x),\ R(\tau,0) = R_1(\tau),\ R_x(\tau,1) = 0,$$

$$x \in (0,1),\ \tau \in (0,d_t), \tag{5}$$

$$R_k^*(\tau) = R(\tau,x_k) + \zeta_k(\tau),$$

(6) where

$$R(T) = 1 + \nu \int_0^T \lambda(T)dT, \quad \zeta_k(\tau) \cong \nu\lambda\left(f_k^*(\tau)\right)\xi_k(\tau),$$

$\widetilde{C}(R) = C(R)/\lambda(R)$, ν, ρ - certain constants. It is necessary to determine $R_0(x)$ and $R_1(\tau)$ using measurement data $R_k^*(\tau)$.

Introduce the following function

$$Q(\tau,x) = \left(\rho\widetilde{C}(R) - 1\right)R_\tau$$

and rewrite (4) as

$$R_\tau - R_{xx} = -Q(\tau,x)$$

(7) Adding equations (5)-(6) we receive a new IHCP formulation where unknown functions are $R_0(x)$, $R_1(\tau)$, as well as $Q(\tau,x)$. These functions are to be determined using observations $R_k^*(\tau)$. Thus on account of increasing a degree of uncertainty the original inverse problem for the quasi-linear heat conduction equation is reformulated into a problem for homogeneous linear heat conduction equation with an extra unknown value - distributed source. The idea of this linearization is propoused by Boguslavsky (1994), when a similar technique is used for linearization of equations of the object and observation as applied to a dynamic system in a state-space. Following the terminology (Boguslavsky, 1994), regard $Q(\tau,x)$ as an uncertain perturbating function.

Since the original problem is ill-posed it should be solved using some additional conditions. If to employ the regularization method and impose some a priori restrictions on the unknown functions, a set of quasi-solutions can be obtained close to the unknown one in this or that sense, and then among those quasi-solutions the best one in the meaning of a desired criterion is to be chosen. Thus, for system (5)-(6), (7) formulate the following optimum control problem: functions $R_0(x)$, $R_1(\tau)$, $Q(\tau,x)$ are to be found minimizing the Tikhonov functional

$$J(\alpha,\theta) = \sum_{i=1}^{N}\int_0^{d_t}\left(R(\tau,x_k) - R_k^*(\tau)\right)^2 d\tau +$$

$$+\alpha^2\ \left\|R_0(x)\right\|^2_{W_2^2} + \left\|R_1(\tau)\right\|^2_{W_2^2} + \theta^2\left\|Q(\tau,x)\right\|^2_{L_2} \quad .$$

(8) With fixed values of α and θ parameters there is the only solution of the problem minimizing (8) (Tikhonov & Arsenin, 1977).

Reduce inverse problem (5)-(6), (7) to finite-dimensional form. Temperature field $R(\tau,x)$ is a superposition of effects of the boundary conditions and the right-hand side of equation (7):

$$R(\tau,x) = \widetilde{R}_1\left(\tau,x,R_0(x)\right) + \widetilde{R}_2\left(\tau,x,R_1(\tau)\right) + \widetilde{R}_3\left(\tau,x,Q(\tau,x)\right).$$

Functions $\widetilde{R}_i\left(\tau,x,f(\tau,x)\right)$ represent a heat conduction equation solution generated by one of functions $R_0(x)$, $R_1(\tau)$, $\widetilde{Q}(\tau,x)$ if the rest are equal to zero. Approximate the unknown dependences by certain systems of basic functions:

$$R_0(x) = \sum_{j=1}^{n1}\beta_j\varphi_j(x), \quad R_1(\tau) = \sum_{j=1}^{n2}\gamma_j\psi_j(\tau),$$

$$Q(\tau,x) = \sum_{j=1}^{n3}\eta_j\phi_j(\tau,x), \tag{9}$$

and also perform time discretization. The corresponding finite-dimensional functional analog (8) can be written as

$$J(\alpha,\theta) = \left\|A\hat{\bar{p}} + B\hat{\bar{\eta}} - \bar{z}\right\|^2_{R^{mN}} +$$

$$+\alpha^2 \left| \left\|F\hat{\bar{p}}\right\|^2_{R^{n1+n2}} + \theta^2\left\|F_3\hat{\bar{\eta}}\right\|^2_{R^{n3}} \right|$$

(10) where

$$i = \overline{1,m}, \quad k = \overline{1,N}, \quad \hat{\bar{p}}^{[n1+n2]} = \left[\bar{\beta}, \quad \bar{\gamma}\right]^T, \quad z_{i\times k} = R_k^*(\tau_i),$$

$$A_{i\times k,j} = \begin{cases} \widetilde{R}_1\left(\tau_i, x_k, \varphi_j(x)\right), & 1 \le j \le n1 \\ \widetilde{R}_2\left(\tau_i, x_k, \psi_j(\tau)\right), & n1+1 \le j \le n1+n2 \end{cases},$$

$$F = \begin{vmatrix} F_1^{[n1\times n1]} & 0 \\ 0 & F_2^{[n2\times n2]} \end{vmatrix},$$

$$B_{i\times k,j} = \widetilde{R}_3\left(\tau_i, x_k, \phi_j(\tau, x)\right), \quad 1 \le j \le n3,$$

$$F_1^T F_1 = \Phi_1, \quad \Phi_1(i,j) = \left\langle \varphi_i(x), \quad \varphi_j(x) \right\rangle_{W_2^2}.$$

(11) Matrices F_2 and F_3 are defined in a similar way (11) with the corresponding basic functions applied.

Compose a system of normal equations for problem (10)

$$\begin{vmatrix} A^T A + \alpha^2 F^T F & A^T B \\ B^T A & B^T B + (\alpha\theta)^2 F_3^T F_3 \end{vmatrix} \begin{bmatrix} \hat{\bar{p}} \\ \hat{\bar{\eta}} \end{bmatrix} = \begin{bmatrix} A^T \bar{z} \\ B^T \bar{z} \end{bmatrix} \qquad (12)$$

and make use the lemma on inversion of block matrix. Since vector $\bar{\eta}$ is of auxiliary character, write an equation to determine the basic solution component $\bar{p}$:

$$\Lambda\hat{\bar{p}} = A^T\bar{z} - A^T B\left(B^T B + (\alpha\theta)^2 F_3^T F_3\right)^{-1} B^T \bar{z}$$

(13) where matrix Λ (the Schur complement) equals to

$$\Lambda = A^T A + \alpha^2 F^T F - A^T B\left(B^T B + (\alpha\theta)^2 F_3^T F_3\right)^{-1} B^T A. \qquad (14)$$

Assume $BF_3^{-1} = U_B S_B V_B^T$ to be a singular decomposition of matrix BF_3^{-1} where U_B, V_B are orthogonal matrices, and $S_B = diag\{s_B\}$. Taking this into account the equation (13) becomes the following

$$\Lambda\hat{\bar{p}} = \left(A^T U_B \widetilde{R}^2 U_B^T A + \alpha^2 F_1^T F\right)\hat{\bar{p}} = A^T U_B \widetilde{R}^2 U_B^T \bar{z} \qquad (15)$$

where

$$\widetilde{R} = diag\{R_i\}, \quad R_i = \begin{cases} (\alpha\theta)\Big/\sqrt{(\alpha\theta)^2 + s_B^2}, & i \le n3 \\ 1, & Mn \ge i > n3 \end{cases} \qquad (16)$$

Note that a system of normal equations (15) corresponds to weighted problem of the least squares.

$$\begin{vmatrix} \widetilde{R} U_B^T A \\ \alpha F \end{vmatrix} \hat{\bar{p}} = \begin{bmatrix} \widetilde{R} U_B^T \bar{z} \\ 0 \end{bmatrix}.$$

ESTIMATION ERRORS AND NUMERICAL RESULTS

Now derive an expression for the estimation error. Let $\bar{\varepsilon} = \hat{\bar{p}} - \bar{p}$. With account of approximation (9) we have

$$\bar{z} = A\bar{p} + B\bar{\eta} + \bar{\zeta} + \bar{\zeta}_a$$

(17) Here ζ_a is an error emerging as a result of the series truncation approximizing the boundary conditions and an uncertain perturbating function. Assume $\zeta_a << \zeta$. Substitute (17) in (15). Then we can write the expression for error

$$\begin{aligned} &\left(A^T U_B \widetilde{R}^2 U_B^T A + \alpha^2 F_1^T F\right)\bar{\varepsilon} = \\ &= \alpha^2 F_1^T F_1 \bar{p} + A^T U_B \widetilde{R}^2 U_B^T B\bar{\eta} + A^T U_B \widetilde{R}^2 U_B^T \bar{\zeta} \end{aligned}. \qquad (18)$$

Then make certain of those parameter values under which the error $\bar{\varepsilon}$ has an acceptable value. If can be seen that $\bar{\varepsilon}$ consists of three components: a bias error $\bar{\varepsilon}_p$, a random error $\bar{\varepsilon}_\zeta$, and an error $\bar{\varepsilon}_\eta$ generated by uncertain perturbating vector $\bar{\eta}$. Write down equation for error $\bar{\varepsilon}_\eta$:

$$\Lambda\bar{\varepsilon}_\eta = A^T\left(U_B \widetilde{R}^2 U_B^T B\right)\bar{\eta}.$$

(19) Consider matrix $U_B \widetilde{R}^2 U_B^T B$. With account of $BF_3^{-1} = U_B S_B V_B^T$ write

$$U_B \widetilde{R}^2 U_B^T B = U_B \widetilde{R}^2 \begin{vmatrix} \widetilde{S}_B^{[n3\times n3]} \\ 0^{[Mn-n3]} \end{vmatrix} V_B^T F_3.$$

With account of (16)

$$\lim_{\theta\to 0}\left\| U_B \widetilde{R}^2 U_B^T B \right\|_E = 0, \quad \partial\left\| U_B \widetilde{R}^2 U_B^T B \right\|_E \Big/ \partial\theta < 0.$$

Therefore, with weighted parameter θ decrease the norm of right-hand side of problem (19) monotonously decreases asymptotically tending to zero, i.e. uncertain perturbating vector $\bar{\eta}$ suppression occurs. But the other estimation error components increase at that caused by $\|\Lambda\|_E$ decrease. It is true, first of all, for bias $\bar{\varepsilon}_p$, the value of which depends on θ only via Λ^{-1}. The behaviour of random error $\bar{\varepsilon}_\zeta$ is determined by interaction of two opposite tendencies, namely: the right-hand side norm decrease and Λ^{-1} operator norm increase. Hence, it is possible to make a conclusion of this approach acceptability in each particular case, first and foremost, on the basis of numerical analysis of estimation error components depending on α and θ parameters. Make similar analysis for pure retrospective IHCP, i.e. such a problem where the unknown is only the initial condition $R_0(x)$. Take eigenfunctions of operator R_{xx} as the basis:

$$R_0(x) = \sqrt{2}\sum_{k=0}^{n} p_k \cos(\pi k x).$$

(20) Suppose interval d_t to be short and the uncertain perturbating function depends mainly on the coordinate, i.e. $Q(\tau, x) \approx Q(x)$. Then approximate $Q(x)$ in the same basis, i.e.

$$Q(x) = \sqrt{2}\sum_{k=0}^{n3} \eta_k \cos(\pi k x).$$

(21) To make the results more instructive consider not $\bar{\varepsilon}_p$, $\bar{\varepsilon}_\eta$ and $\bar{\varepsilon}_p$ which are vectors but some integral values. For

example, for the bias characteristic caused by vector $\overline{p}$ i-component presence introduce coefficients $K_{1,i}$ such as

$$\sqrt{2}\int_0^1 \left| \sum_{k=0}^{n} \varepsilon_{p,k} \cos(\pi k x) \right| dx = K_{1,i} |p_i|$$

and for the estimation error component characteristic generated by uncertain perturbating vector $\overline{\eta}$ i-component apply coefficients $K_{2,i}$ such as

$$\sqrt{2}\int_0^1 \left| \sum_{k=0}^{n} \varepsilon_{\eta,k} \cos(\pi k x) \right| dx = K_{2,i} |\eta_i| .$$

For the random estimation error component characteristic use coefficient K_3 such as

$$\int_0^1 \left\| \sqrt{2} \sum_{k=1}^{n} \varepsilon_{\zeta,k} \cos(\pi k x) \right\|_E dx = K_3 \sigma$$

where

$$\sigma^2 = M\left[\overline{\zeta}^T \overline{\zeta}\right].$$

The results of calculating coefficients $\lg K_{1,i}$, $\lg K_{2,i}$, K_3 depending on $\lg\theta$ for different values of the regularization parameter α are shown in Fig.1,2,3. The j-curve corresponds to value $\lg\alpha = -4.0 + (j-1)/3$. The observations were supposed to take place at three points with coordinates $x_k = \{0.2,\ 0.6,\ 1.0\}$, estimation interval $d_t = 0.08$, observation number within the interval $m = 24$, approximation orders $n = n3 = 8$.

Consider a specific example. Assume the unknown initial distribution to contain only one expansion component (20) with a number $k = 1$. Suppose $\alpha = 0.01$. This case corresponds to curves in the charts indicated by number 7. As seen in Fig.2, $K_3 \cong 1$ i.e. the random estimation error component has the order of instrumental error. With decrease of weighted parameter θ from 10^4 to 10^1 the bias increases (Fig.1) twice as much, however, its absolute value does not exceed a tenth portion of percent in the result. The effect of the first four expansion components (21) of uncertain perturbating function (Fig.3) decreases, approximately, in the following proportion: 1/15, 1/9, 1/6. 1/3.

CONCLUSION

Thus, original IHCP (4)-(6) is nonlinear. We interprete it as an inverse problem for a linear system having an extra input $Q(\tau, x)$. On account of including $Q(\tau, x)$ into the unknown vector-function structure the estimation system is built, the sensitivity of which for this input is controlled by parameter θ. In a number of cases the $Q(\tau, x)$ effect upon the result can be suppressed under the condition that the other estimation error components are within the acceptable range.

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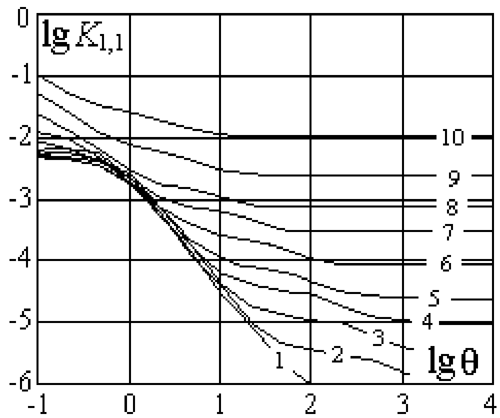


Fig. 1. The gain factor $K_{1,1}$ (bias) upon parameters θ and α.

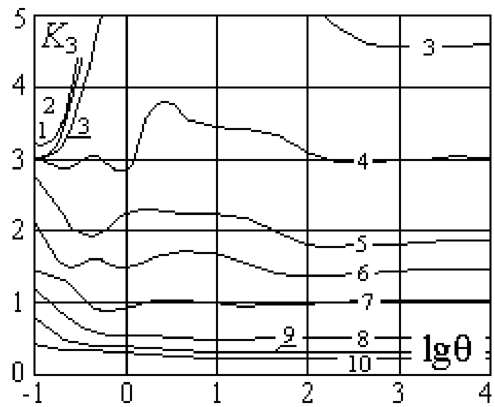
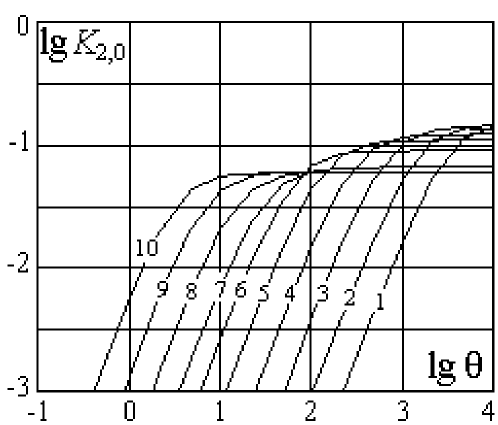
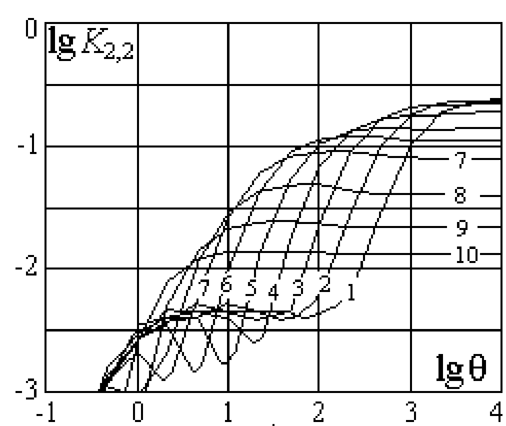


Fig.2. The gain factor K_3 (random error) upon parameters θ and α.



a)



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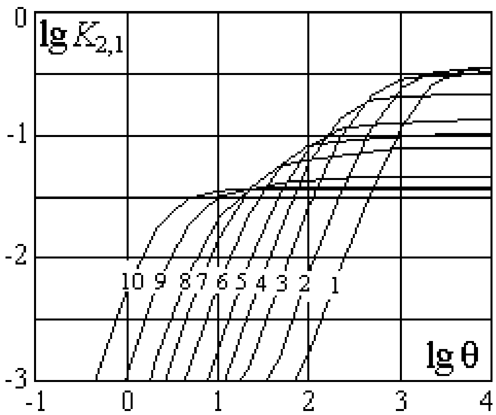

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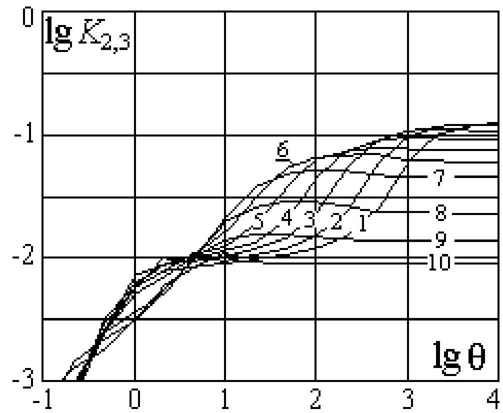

d)

Fig 3(a-d). The gain factors $K_{2.,0}$, $K_{2,1}$, $K_{2,2}$, $K_{2,3}$ (the error generated by uncertain perturbating vector) upon the parameters θ and α.