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OPTIMIZATION OF MEASUREMENTS FOR INVERSE PROBLEM

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ABSTRACT

A new method for obtaining the optimal experimental/measurement procedure for general estimation problems is proposed. This approach is based on the framework of the Kalman filter technique. The eigen values of a posteriori estimate error covariance matrix depend on measurement conditions, such as geometries of specimens, structure of experimental sets, locations of measurements and types of measurements. The optimal measurement condition is obtained by minimizing the maximum eigen value of a posteriori estimate error covariance matrix. Candidates of measurement condition are firstly put up, and combinational optimization method is carried out. Some examples are presented to demonstrate that the present method could be utilized effectively for designing estimation system.

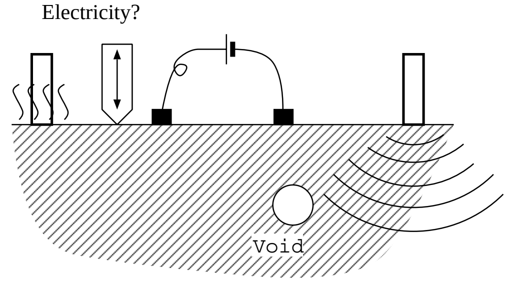

Figure 1. VARIETY OF MEASUREMENT SETTINGS

INTRODUCTION

Inverse Problems can be found in many topics of engineering. Generally speaking, solution of an inverse problem entails determining unknown causes based on observation of their effects. Many researches have been done to overcome ill-conditioned problems (Trujillo,1997) (Engl,1996) (Hensel,1991).

However, not many researches have been done on a guideline how to collect measurement data to relieve ill-condition. For example, let's consider an inverse problem of the nondistructive void location detection(see Figure 1). Following questions would be arise.

- What kind of physical phenomena is the best to apply? (elasto-dynamics, elasto-statics, electricity, thermal transfer, etc.)
- What kind of physical quantity is the best to measure? (displacement, strain, force, acceleration, potential, current, temperature, etc.)
- Where is the best location for measurement?
- When is the best timing for measurement?

In this paper, a new method for obtaining the optimal **measurement condition** for general estimation problems is proposed. Some examples are presented to demonstrate that the present method could be utilized effectively for designing estimation system.

THEORETICAL GROUNDWORK

Generally, the discretized inverse problem can be represented as following formula,

$$\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}) + \boldsymbol{w} \tag{1}$$

where, $\boldsymbol{y}$ is the m dimensional observation vector, $\boldsymbol{x}$ is the n dimensional vector to be estimated, $\boldsymbol{h}$ is the non-linear m dimensional function which represents the problem. The random variable $\boldsymbol{w}$ represent the measurement noise. It is assumed to be independent, white, and with normal probability distribution

$$p(\boldsymbol{w}) = N(\bar{\boldsymbol{w}}, W). \tag{2}$$

We define $\bar{\boldsymbol{x}}$ to be our a priori estimate before observing any data, and $\hat{\boldsymbol{x}}$ to be our a posteriori estimate after given measurement. We can then define a priori and a posteriori estimate errors as

$$\bar{\boldsymbol{e}} \equiv \boldsymbol{x} - \bar{\boldsymbol{x}} \tag{3}$$
$$\hat{\boldsymbol{e}} \equiv \boldsymbol{x} - \hat{\boldsymbol{x}} \tag{4}$$

The a priori estimate error covariance M is then

$$M = E[\ \bar{\boldsymbol{e}}\ \bar{\boldsymbol{e}}^t]. \tag{5}$$

The a priori probability density function of $\boldsymbol{x}$ is

$$p_1(\boldsymbol{x}) = \frac{1}{\sqrt{(2\pi)^3|M|}} exp\left\{-\frac{1}{2}(\boldsymbol{x}-\bar{\boldsymbol{x}})^t M^{-1}(\boldsymbol{x}-\bar{\boldsymbol{x}})\right\}. \tag{6}$$

The probability density function of $\boldsymbol{w}$ is

$$p_2(\boldsymbol{w}) = \frac{1}{\sqrt{(2\pi)^n|W|}} exp\left\{-\frac{1}{2}(\boldsymbol{w}-\bar{\boldsymbol{w}})^t W^{-1}(\boldsymbol{w}-\bar{\boldsymbol{w}})\right\}. \tag{7}$$

The conditional probability of $\boldsymbol{y}$ given $\boldsymbol{x}$ is

$$\begin{aligned} p_2(\boldsymbol{y}|\boldsymbol{x}) &= \frac{1}{\sqrt{(2\pi)^n|W|}} \times \\ & exp\left\{-\frac{1}{2}(\boldsymbol{y}-h(\boldsymbol{x})-\bar{\boldsymbol{w}})^t W^{-1}(\boldsymbol{y}-h(\boldsymbol{x})-\bar{\boldsymbol{w}})\right\} \end{aligned} \tag{8}$$

The probability of $\boldsymbol{y}$ is

$$\begin{aligned} p_3(\boldsymbol{y}) &= \frac{1}{\sqrt{(2\pi)^n|W+HMH^t|}} \times \\ & exp\left\{-\frac{1}{2}(\boldsymbol{y}-\bar{\boldsymbol{y}})^t (W+HMH^t)^{-1}(\boldsymbol{y}-\bar{\boldsymbol{y}})\right\}, \end{aligned} \tag{9}$$

where H is the Jacobian matrix of $h()$ which is given as the following(Arimoto,1992),

$$H_{ij} = \frac{\partial h_i}{\partial x_j}\bigg|_{\boldsymbol{x}=\tilde{\boldsymbol{x}}} \tag{10}$$

Applying the following Bayes theorem:

$$p(\boldsymbol{x}|\boldsymbol{y}) = \frac{p_1(\boldsymbol{x})p_2(\boldsymbol{y}|\boldsymbol{x})}{p_3(\boldsymbol{y})}, \tag{11}$$

the conditional probability of $\boldsymbol{x}$ given $\boldsymbol{y}$ is

$$p(\boldsymbol{x}|\boldsymbol{y}) = \frac{1}{\sqrt{(2\pi)^2|P|}} exp\left\{-\frac{1}{2}(\boldsymbol{x}-\hat{\boldsymbol{x}})^t P^{-1}(\boldsymbol{x}-\hat{\boldsymbol{x}})\right\}, \tag{12}$$

where P and $\hat{\boldsymbol{x}}$ are given as the following (Arimoto,1992)

$$\hat{\boldsymbol{x}} = \bar{\boldsymbol{x}} + PH^tW^{-1}(\bar{\boldsymbol{y}} - \boldsymbol{h}(\tilde{\boldsymbol{x}}) - \bar{\boldsymbol{w}}) \tag{13}$$
$$P = (M^{-1} + H^tW^{-1}H)^{-1} \tag{14}$$

Therefore, the a posteriori estimate error covariance is

$$P \equiv E[\ \hat{\boldsymbol{e}}\ \hat{\boldsymbol{e}}^t] = (M^{-1} + H^tW^{-1}H)^{-1}. \tag{15}$$

OPTIMIZATION OF MEASUREMENT

In order to express the function $\boldsymbol{h}()$ depends on the measurement condition explicitly, let's represent eqn.(1) as the following,

$$\boldsymbol{y} = \boldsymbol{h}_{\mathcal{M}}(\boldsymbol{x}) + \boldsymbol{w}_{\mathcal{M}} \tag{16}$$

where $\mathcal{M}$ is conceptual notation for the measurement condition. Equation (15) also expressed with this notation.

$$P_{\mathcal{M}} \equiv E[\ \hat{\boldsymbol{e}_{\mathcal{M}}}\ \hat{\boldsymbol{e}_{\mathcal{M}}}^t] = (M^{-1} + H_{\mathcal{M}}^t W_{\mathcal{M}}^{-1} H_{\mathcal{M}})^{-1}. \tag{17}$$

where the subscript $\mathcal{M}$ denotes that the value depend on the measurement condition.

Our problem here is to find the best measurement condition which gives the most accurate estimation in the inverse problem. The accuracy of the estimation can be indicated with the error covariance $P_{\mathcal{M}}$. Figure 2 shows the visible meaning of $P_{\mathcal{M}}$ in two dimensional estimate space, where λ_1 and λ_2 are the eigen values of $P_{\mathcal{M}}$. The smaller

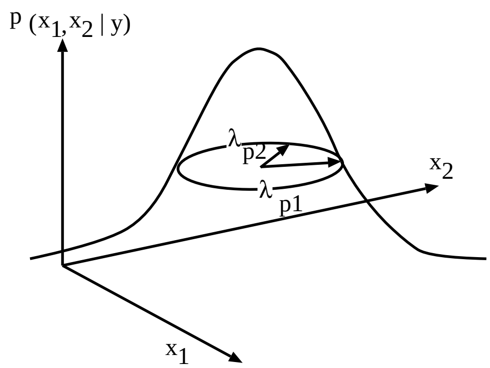

Figure 2. PROBABILITY DENSITY FUNCTION FOR ESTIMATES X

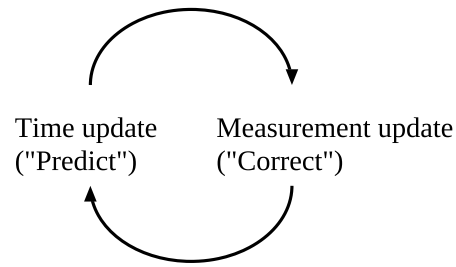

Figure 3. THE ONGOING DISCRETE KALMAN FILTER CYCLE

the eigen values λ_i become, the more accurate the estimation becomes. Therefore, our problem reduced to solving the following combinational/discrete optimization problem

$$\min_{arg=\mathcal{M}}((\lambda_{max}(P_{\mathcal{M}}))^2) \qquad (\mathcal{M} \subset \mathcal{U}) \tag{18}$$

where $\lambda_{max}(P)$ is the maximum eigen value of P, U is the universal set consists of all candidates of measurement condition. Above optimization problem can be solved with several method such as genetic algorithm, integer programming.

APPLICATION FOR DYNAMIC SYSTEM

In this section, we consider the application for dynamic data collection. In this case, present method is very suitable to Kalman filter algorithm(Kalman,1960) (Gelb,1974). Let us assume that our process has a state vector $\boldsymbol{x}$ and the process is now governed by the stochastic difference equation

$$\boldsymbol{x}_{k+1} = \boldsymbol{f}_k(\boldsymbol{x}_k) + \boldsymbol{v}_k. \tag{19}$$

where the random variable $\boldsymbol{v}_k$ represent the process noise. The non-linear function $\boldsymbol{f}(\cdot)$ relates the state at time step k to the state at step $k+1$.

The Kalman filter estimates a process by using a form of feedback control: the filter estimates the process state at some time and then obtains feedback in the form of (noisy) measurements. As such, the equations for the Kalman filter fall into two groups: time update equations and measurement update equations.

The time update equations are responsible for projecting forward (in time) the current state and error covariance estimates to obtain the a priori estimates for the next time step. The measurement update equations are responsible for the feedback i.e. for incorporating a new measurement into the a priori estimate to obtain an improved a posteriori estimate.

The time update equations can also be thought of as predictor equations, while the measurement update equations can be thought of as corrector equations. Indeed the final estimation algorithm resembles that of a predictor-corrector algorithm for solving numerical problems as shown in Figure 3.

The specific equations for the time updates are presented below.

$$\bar{\boldsymbol{x}}_{k+1} = \boldsymbol{f}_k(\hat{\boldsymbol{x}}_k) + \bar{\boldsymbol{v}} \tag{20}$$
$$M_{k+1} = F_k P_k F_k^t + V_k \tag{21}$$

where $F_k = \partial \boldsymbol{f}_k / \partial \boldsymbol{x}$. The specific equations for the measurement updates are presented below.

$$K_k = M_k H_k^t (H_k M_k H_k^t + W_k)^{-1} \tag{22}$$
$$P_k = (I - K_k H_k) M_k \tag{23}$$
$$\hat{\boldsymbol{x}}_k = \bar{\boldsymbol{x}}_k + K_k(\boldsymbol{y}_k - h_k(\bar{\boldsymbol{x}}_k)) \tag{24}$$

Again notice how the time update equations in eqn.(20) and (21) project the state and covariance estimates from time step k to step k+1.

The first task during the measurement update is to compute the Kalman gain, K_k. The next step is to actually measure the process to obtain $\boldsymbol{y}_k$, and then to generate an a posteriori state estimate by incorporating the measurement as in (24). The final step is to obtain an a posteriori error covariance estimate via (23).

After each time and measurement update pair, the process is repeated with the previous a posteriori estimates used to project or predict the new a priori estimates. This

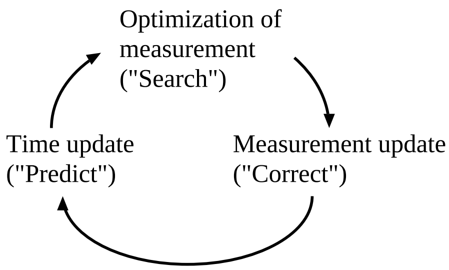

Figure 4. IMPLEMENTATION OF MEASUREMENT OPTIMIZATION TO KALMAN FILTER CYCLE

recursive nature is one of the very appealing features of the Kalman filter it makes practical implementations much feasible.

In our aproach, we consider to optimize the measurement condition for each time step k. Eqn. 22 and 23 can be represent as the following to show that these matrix depend on measurement condition:

$$K_{\mathcal{M}k} = M_k H^t_{\mathcal{M}k}(H_{\mathcal{M}k} M_k H^t_{\mathcal{M}k} + W_{\mathcal{M}k})^{-1} \tag{25}$$

$$P_{\mathcal{M}k} = (I - K_{\mathcal{M}k} H_{\mathcal{M}k}) M_k \tag{26}$$

Different $P_{\mathcal{M}k}$ will be calculated for each different obeservation condition, and each $P_{\mathcal{M}k}$ has different values of eigen values. In order to get the optimum measurement condition, the followinig optimization problem will be solved:

$$\min_{arg=\mathcal{M}}((\lambda_{max}(P_{\mathcal{M}k}))^2) \qquad (\mathcal{M} \subset \mathcal{U}). \tag{27}$$

After getting the optimum mesurement condition $\boldsymbol{h}_{\mathcal{O}k}(\cdot)$, where $\mathcal{O}$ is the supscript for optimum solution, the measurement process to obtain $\boldsymbol{y}_k$ is performed. The equation to generate an a posteriori state estimate is:

$$\hat{\boldsymbol{x}}_k = \bar{\boldsymbol{x}}_k + K_{\mathcal{O}k}(\boldsymbol{y}_{\mathcal{O}k} - h_{\mathcal{O}k}(\bar{\boldsymbol{x}}_k i)) \tag{28}$$

The numerical algorithm which implements the optimization of measurement condition can be shown as Figure 4

EXAMPLE ANALYSIS

Estimation of Gurson's material parameters

In this section, measurement for estimation of Gurson's material parameters will be optimized. Gurson's constitutive model has been widely used for studying ductile fracture as well (Gurson,1977). Accurate determination of their parameters is important since incorrect values will lead to erroneous results in simulation analyses.(Aoki,1997)

The flow potential of this Gurson's material is expressed as,

$$\Phi = \frac{\sigma_e^2}{\overline{\sigma}^2} + 2f\cosh\left(\frac{3\sigma_h}{2\overline{\sigma}^2}\right) - (1+f)^2 = 0 \tag{29}$$

Here σ_e is the effective stress, σ_h is the hydrostatic stress, $\overline{\sigma}$ is the current tensile flow stress of the matrix material, f is the void volume fraction. The void volume fraction f increases by void nucleation as well as void coalescence or growth. The rate of increase can be decomposed as $\dot{f} = \dot{f}_{\text{nucl}} + \dot{f}_{\text{grow}}$. Here $\dot{(\;)}$ represents the time derivative. The second term in the RHS of the above equation represents the increase of the void volume fraction due to plastic deformation and it can be derived as $\dot{f}_{\text{grow}} = (1-f)D^p_{kk}$, where D^p_{kk} denotes the plastic part of the rate of deformation tensor relating to dilatational change of the porous medium. The rate of void volume increase due to nucleation is also estimated as

$$\dot{f}_{nucleation} = A\dot{\varepsilon}^p_m \tag{30}$$

where

$$A = \frac{f_N}{s_N\sqrt{2\pi}} exp\left\{-\frac{(\varepsilon^p_m - \varepsilon_N)^2}{2s_N^2}\right\}. \tag{31}$$

In the procedure, we assign $\boldsymbol{x} \equiv (f_N, \varepsilon_N, S_N)^t$ to be the vector containing the unknown parameters.

Following items are considered for the measurement condition.

- Specimen geometry: smooth bar type or notched bar type.
- Types of measurements: strain, load or displacements.
- Location of measurement
- Timing of measurement: initial state or latter state (See Figure 7).

For the specimen models, smooth bar type and notched bar type, as shown in Figure 5, are considered. In our method, the required quantities are the matrix relating to the measurement error $W_{\mathcal{M}}$ the initial estimate $\bar{\boldsymbol{x}}$, the initial covariant matrix M, and the calculated derivative $H_{\mathcal{M}}$ for all candidates of measurement condition. FEM Analysis was performed to get $H_{\mathcal{M}}$ as the following:

$$H_{\mathcal{M}ij} \equiv \frac{\partial h_{\mathcal{M}i}}{\partial x_j}\Big|_{\boldsymbol{x}=\bar{\boldsymbol{x}}} \approx \frac{h_{\mathcal{M}i}(\bar{\boldsymbol{x}} + \Delta\boldsymbol{x}) - h_{\mathcal{M}i}(\bar{\boldsymbol{x}})}{\Delta x_j} \tag{32}$$

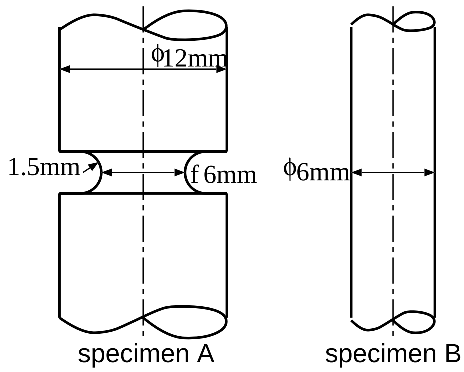

Figure 5. GEOMETRY OF SPECIMEN

where, $\bar{\boldsymbol{x}} = (f_n, \varepsilon_N, S_N) = (0.05, 0.1, 0.1)$ and $\Delta x_1 = \Delta f_n = 0.005$, $\Delta x_2 = \Delta \varepsilon_N = 0.01$, $\Delta x_2 = \Delta \S_N = 0.01$. The material constants are chosen as $E = 207[GPa]$ and $\nu = 0.333$ in the constitutive equation.

The FEM mesh is shown in Figure 6 with candidates of measurement locations on. The number of possible measurements is 224 which are combinations of geometory of specimen, measurement type, location and timing. Note that the number of elements in U in eqn.(18) is ${}_{224}C_m$, where m is the number of allowed measurement.

Measurement errors which relate to $W_{\mathcal{M}}$ are set as the following:

Displacement:	$\sigma = 5[\mu m]$
Strain:	$\sigma = 1\%$ of actual value
Load:	$\sigma = 10[kg]$

where σ is the standard deviation of the error. The optimum condition was obtained by making a thorough search for this example.

Figure 8 shows the estimation error for each allowed number of measurement. The dashed line is a result using only specimen A (notched bar), chained line is a result using only specimen B (smooth bar) and solid line is a result using both specimens. It is seen that the estimation error decreased as many measurements are allowd. In order to achieve an error less than 5%, more than 10 measurements are required.

Table 1 shows the best ranking of selected measurements, where A,B,C and D on the location column indicates the location of measurements which is shown in Figure 9. For example, if only 4 measurements are allowed, the combination of top 4 measurements on this table give the minimun estimation error.

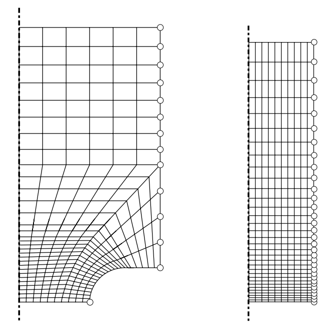

Figure 6. FEM MESH AND MEASUREMENT LOCATIONS

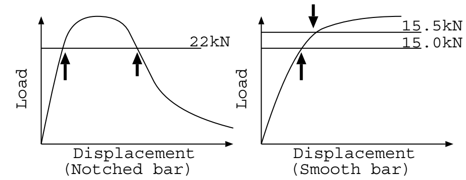

Figure 7. TWO KINDS OF LOAD CONDITION

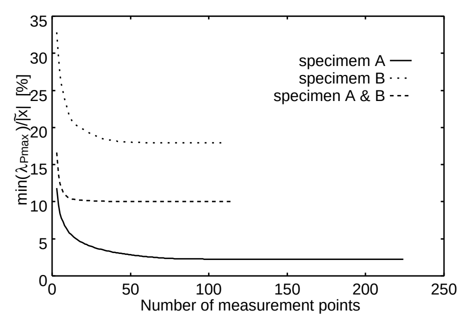

Figure 8. ESTIMATION ERROR VS. NUMBER OF MEASUREMENT

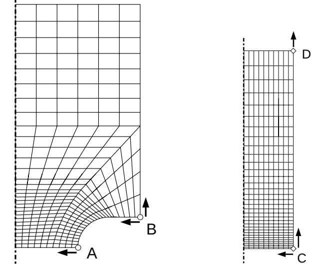

Figure 9. OPTIMIZED MEASUREMENT LOCATION (NOTCHED BAR SPECIMEN)

Table 1. BEST RANKING OF MEASUREMENTS

Ranking	Type	Load	Specimen	Location
1	ε_θ	15.5KN	smooth	C
2	ε_θ	15.0KN	smooth	C
3	ε_z	12.0KN(2nd)	notched	B
4	ε_z	15.0KN	smooth	C
5	ε_θ	12.0KN(2nd)	notched	B
6	ε_θ	12.0KN(1st)	notched	B
7	ε_z	15.5KN	smooth	C
8	ε_z	12.0KN(1st)	notched	B
9	u_z	12.0KN(2st)	notched	B
10	u_z	15.0KN	smooth	D
11	u_z	15.5KN	smooth	D
12	u_r	12.0KN(1st)	notched	A
13	u_z	12.0KN(1st)	notched	B
14	u_r	15.0KN	smooth	C
15	u_r	15.5KN	smooth	C
16	u_r	12.0KN(2nd)	notched	A
17	u_r	12.0KN(2nd)	notched	B
18	u_r	12.0KN(1st)	notched	B

CONCLUSION

In this paper a new method for obtaining the optimal experimental/measurement procedure for general estimation problems is proposed. This approach is based on a framework of the Kalman filter technique. The eigen values of a posteriori estimate error covariance matrix depend on measurement conditions, such as geometries of specimens, structure of experimental sets, locations of measurements and types of measurements. The optimal measurement condition is obtained by minimizing the maximum eigen value of a posteriori estimate error covariance matrix. Candidates of measurement condition are firstly put up, and combinational optimization method is carried out. Some examples are presented to demonstrate that the present method could be utilized effectively for designing estimation system.

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