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A BEM- AND ADJOINT VARIABLE-BASED APPROACH TO CRACK SHAPE SENSITIVITY ANALYSIS

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Abstract

This communication addresses a computation strategy, based on the adjoint variable approach and BIE/BEM formulations of the direct problem, for evaluating crack or void shape sensitivities of objective functionals. Boundary-only expressions for such sensitivities are sought, in the context of linear elastodynamics.

In the case of a void, boundary-only expressions for sensitivities of integral functionals defined on (part of) the external boundary are easy to obtain by the standard adjoint variable approach. When the void degenerates to a crack, the previous result ceases to be applicable, however, because non-integrable terms arise due to crack-tip singularities.

We show, for two classes of crack perturbations, that boundary-only sensitivity expressions using an adjoint state can still be obtained: (1) simple transformations (translation, rotation or expansion of the crack) of arbitrarily shaped domains, and (2) general two-dimensional geometries and crack perturbations. In the latter case, the shape sensitivity is expressed using the primal and adjoint stress intensity factors.

Numerical tests of the latter kind of sensitivity expression are presented for a 2-D body with an internal crack, in plane-strain elastodynamics. The influence of crack shape perturbations on an objective functional is examined. The sensitivity results obtained using the present strategy compare well with finite difference evaluations.

1 Introduction

The need to compute the sensitivity of integral functionals with respect to shape parameters arises in many situations where a geometrical domain plays a primary role; shape optimization and inverse problems are the most obvious, as well as possibly the most important, of such instances. In addition to numerical differentiation techniques, shape sensitivity evaluation can be based on either direct differentiation or the adjoint variable approach, this paper being focused on the latter. Besides, consideration of shape changes in otherwise linear problems makes it very attractive to use boundary integral equation (BIE) formulations, which constitute the minimal modelling as far as the geometrical support of field variables is concerned.

In the BIE context, the direct differentiation approach relies upon the material differentiation of the governing BIEs in either singular form (Barone and Yang, 1989; Mellings and Aliabadi, 1995) or regularized form (Bonnet, 1995b; Matsumoto et al., 1993; Nishimura, 1995). The usual material differentiation formula for surface integrals is shown to be applicable to strongly singular or hypersingular BIEs as well (Bonnet, 1997). Thus, the direct differentiation approach is in particular applicable in the presence of cracks. Following this approach, a shape sensitivity computation relies on solving as many new boundary-value problems as the numbers of shape parameters present. Since they all involve the same, original, governing operator, the computational effort is reduced to setting up new right-hand sides and solving new linear systems by backsubstitution.

The adjoint variable approach is even more attractive, since it needs to solve only one new boundary-value problem (the adjoint problem) per integral functional present (often only one), whatever the number of shape parameters. In connexion with BIE formulations alone, the adjoint variable approach has been successfully applied to many shape sensitivity problems, cf. e.g. Aithal and Saigal, 1995; Bonnet, 1995a; Burczyński, 1993; Burczyński et al., 1995; Choi and Kwak, 1988; Meric, 1995. This

relies heavily upon the possibility of formulating the final, analytical expression of the shape sensitivity of a given integral functional as a boundary integral that involves the boundary traces of the primary and adjoint solutions. However, when the geometrical domain under consideration contains cracks or other geometrical singularities, divergent integrals associated with e.g. crack tip singularity of field variables arise, and obtaining a boundary-only sensitivity expression raises mathematical difficulties.

The present paper deals with the formulation of the adjoint variable method applied to crack shape sensitivity analysis, in connexion with the use of BIE formulations for elastodynamics in the time domain. The corresponding boundary-only formula for the shape sensitivity of the functional is first established for the case of an unknown void. It is then shown to become inconsistent in the limit when the void becomes a crack because of the divergence of a certain domain integral. However, resting on the analysis made for the case of a void, functional shape sensitivity expressions consistent with the use of BIE formulations and applicable to crack identification problems are derived for two cases. Firstly, simple shape transformations (translations, rotations, expansion) can be considered for either 2-D or 3-D problems. Secondly, a sensitivity formula involving integrals on the crack and on arbitrary contours around the crack tips are established for 2-D situations. They hold regardless of the crack shape and of the shape transformation.

2 Motivation for shape sensitivity analysis

Consider a bounded domain B with external boundary S which contains an internal defect in the form of either a void V of boundary Γ (Fig. 1a) or a crack with crack surface Γ (Fig. 1b). Let Ω denote the actual body (i.e.containing the defect): $\Omega = B \setminus V$ or $\Omega = B \setminus \Gamma$. The displacement $\boldsymbol{u}$, strain $\boldsymbol{\varepsilon}$ and stress $\boldsymbol{\sigma}$ are related by the well-known field equations of linear elastodynamics in the time domain ($\boldsymbol{C}$: fourth-order elasticity tensor):

$$\begin{aligned} &\mathrm{div}\boldsymbol{\sigma} - \rho\ddot{\boldsymbol{u}} = \boldsymbol{0} \\ &\boldsymbol{\sigma} = \boldsymbol{C}:\boldsymbol{\varepsilon} \qquad\qquad \text{in } \Omega \\ &\boldsymbol{\varepsilon} = \tfrac{1}{2}(\boldsymbol{\nabla u} + \boldsymbol{\nabla}^T\boldsymbol{u}) \end{aligned} \tag{1}$$

The shape and position of the boundary Γ characterizing the defect are unknown. Suppose that a given traction $\bar{\boldsymbol{f}}$ is imposed on

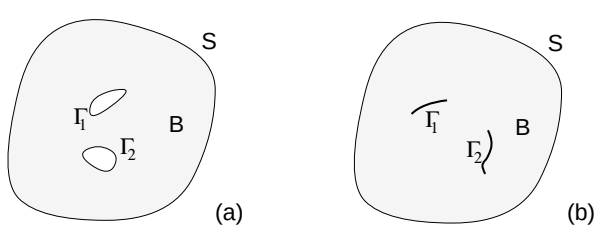

Figure 1. A body with an internal defect: (a) void, (b) crack

S, while Γ is traction-free. Assuming initial rest, equations (1) are completed by the following boundary and initial conditions:

$$\begin{aligned} &\boldsymbol{f} = \bar{\boldsymbol{f}} && \text{on } S \\ &\boldsymbol{f} = \boldsymbol{0} && \text{on } \Gamma \\ &\boldsymbol{u} = \dot{\boldsymbol{u}} = \boldsymbol{0} && \text{in } \Omega, \text{ at } t = 0 \end{aligned} \tag{2}$$

where $\boldsymbol{f} \equiv \boldsymbol{\sigma}.\boldsymbol{n}$ is the traction vector, defined in terms of the outward unit normal $\boldsymbol{n}$ to Ω. In the case of a crack, the displacement $\boldsymbol{u}$ is allowed to jump across Γ; $[\![\boldsymbol{u}]\!] \equiv \boldsymbol{u}^+ - \boldsymbol{u}^- \neq \boldsymbol{0}$.

Consider the problem of finding the shape and position of the defect using elastodynamic experimental data, as in ultrasonic measurements. The lack of information about V and Γ is compensated by some knowledge about $\boldsymbol{u}$ on S (redundant boundary data). Assume that a measurement $\hat{\boldsymbol{u}}(\boldsymbol{x},t)$ of $\boldsymbol{u}$ is available for $\boldsymbol{x} \in S_m \subseteq S$ and $t \in [0,T]$. The usual approach for finding Γ is the minimization of some distance J between $\boldsymbol{u}_\Gamma$ (computed) and $\hat{\boldsymbol{u}}$ (measured), e.g.:

$$J(\Gamma) = \frac{1}{2}\int_0^T \int_{S_m} |\hat{\boldsymbol{u}} - \boldsymbol{u}_\Gamma|^2 \, \mathrm{d}S\,\mathrm{d}t \tag{3}$$

where $\boldsymbol{u}_\Gamma$ denotes the solution of problem (1, 2) for a given location of Γ. The minimization of J with respect to Γ needs in turn, for efficiency, the evaluation of the functional J and its gradient with respect to perturbations of Γ.

Other kinds of sensitivity problems with different motivations (e.g. optimization) can be considered as well. Let us thus introduce the following generic objective function:

$$J(\Gamma) = \int_0^T \int_S \varphi(\boldsymbol{u}_\Gamma, \boldsymbol{x}, t)\,\mathrm{d}S\,\mathrm{d}t + \int_\Gamma \psi(\boldsymbol{x})\,\mathrm{d}S \tag{4}$$

Because of the fact that functionals J of the type (4) depend only on boundary quantities and the problem (1, 2) does not involve sources distributed over the domain Ω then it is natural to solve the direct problem by means of the boundary element method.

3 Boundary integral equation for the direct problem

BIE formulations for the direct elastodynamic problem are based on either a dynamic fundamental solution (BIE formulation in the time or frequency domain) or a static (time-independent) fundamental solution together with the dual reciprocity method (DRM). In the latter, which has been used in our numerical experiments so far, the acceleration within the domain Ω is approximated by a set of A given co-ordinate functions $r^\alpha(\boldsymbol{y})$:

$$\ddot{\boldsymbol{u}}(\boldsymbol{y},t) = \sum_{\alpha=1}^{A} \ddot{s}^\alpha(t)\boldsymbol{r}^\alpha(\boldsymbol{y}) \tag{5}$$

where $s^\alpha(t)$ is a set of unknown, time dependent, functions. The boundary integral equation takes the form:

$$\frac{1}{2}\boldsymbol{u}(\boldsymbol{x},t) + \fint_{\partial\Omega} \boldsymbol{T}(\boldsymbol{x},\boldsymbol{y}).\boldsymbol{u}(\boldsymbol{y},t)\,dS_y - \int_{\partial\Omega} \boldsymbol{U}(\boldsymbol{x},\boldsymbol{y}).\boldsymbol{f}(\boldsymbol{y},t)\,dS_y$$

$$= \sum_{\alpha=1}^{A} \left\{ \frac{1}{2}\tilde{\boldsymbol{u}}^{\alpha}(\boldsymbol{x}) + \fint_{\partial\Omega} \boldsymbol{T}(\boldsymbol{x},\boldsymbol{y}).\tilde{\boldsymbol{u}}^{\alpha}(\boldsymbol{y})\, dS_y \right.$$
$$\left. - \int_{\partial\Omega} \boldsymbol{U}.(\boldsymbol{x},\boldsymbol{y})\tilde{\boldsymbol{f}}^{\alpha}(\boldsymbol{y})\, dS_y \right\} \ddot{s}^{\alpha}(t) \qquad (6)$$

where $\boldsymbol{U}(\boldsymbol{x},\boldsymbol{y})$ and $\boldsymbol{T}(\boldsymbol{x},\boldsymbol{y})$ are the elastostatic fundamental displacement and traction (usually associated with the Kelvin solution), and $(\tilde{\boldsymbol{u}}^{\alpha}(\boldsymbol{y},t), \tilde{\boldsymbol{f}}^{\alpha}(\boldsymbol{y},t)$ are the displacement and traction generated by the body force $\boldsymbol{r}^{\alpha}(\boldsymbol{y})$. To solve the direct elastodynamic problem using the DRM the boundary is divided into boundary elements, and displacements $\boldsymbol{u},\tilde{\boldsymbol{u}}$ and tractions $\boldsymbol{f},\tilde{\boldsymbol{f}}$ within each element are approximated using the same interpolation functions. As a result, a system of ordinary differential equations in time is obtained.

Equation (6) can be applied to the void problem. When used for the crack problem with collocation points $\boldsymbol{x}$ on both crack faces $\Gamma^{\pm}$, the resulting set of integral equations becomes singular. In order to overcome this problem without introducing domain subdivisions, which are not convenient in geometrical inverse problems, equation (6) is replaced on one crack face by a new, independent, traction integral equation; this is the so-called dual formulation. The traction equation for the DRM has the form (Fedelinski et al., 1996):

$$\frac{1}{2}\boldsymbol{f}(\boldsymbol{x},t) + \fint_{\Gamma} \boldsymbol{D}(\boldsymbol{x},\boldsymbol{y}).\boldsymbol{u}(\boldsymbol{y},t)\, dS_y$$
$$+ \int_{S} \left[\boldsymbol{D}(\boldsymbol{x},\boldsymbol{y}).\boldsymbol{u}(\boldsymbol{y},t) - \boldsymbol{T}(\boldsymbol{y},\boldsymbol{x}).\boldsymbol{f}(\boldsymbol{y},t) \right] dS_y$$
$$= \sum_{\alpha=1}^{A} \left\{ \frac{1}{2}\tilde{\boldsymbol{f}}^{\alpha}(\boldsymbol{x}) + \fint_{\Gamma} \boldsymbol{D}(\boldsymbol{x},\boldsymbol{y})\tilde{\boldsymbol{u}}^{\alpha}(\boldsymbol{y})\, dS_y \right.$$
$$\left. + \int_{S} \left[\boldsymbol{T}(\boldsymbol{y},\boldsymbol{x})\tilde{\boldsymbol{u}}(\boldsymbol{y}) - \boldsymbol{T}(\boldsymbol{y},\boldsymbol{x})\tilde{\boldsymbol{f}}(\boldsymbol{y}) \right] dS_y \right\} \ddot{s}^{\alpha}(t) \qquad (7)$$

where $\boldsymbol{D}(\boldsymbol{x},\boldsymbol{y}) = [\boldsymbol{C}:\nabla_x(\boldsymbol{C}:\nabla_y \boldsymbol{U}(\boldsymbol{x},\boldsymbol{y}))].\boldsymbol{n}$ is the hypersingular static kernel asssociated with the traction integral equation, while the symbols $\fint$ and $\fint$ denote singular integrals of the Cauchy principal value and Hadamard finite part types, respectively. Equation (7) can also be used with $\boldsymbol{x}$ on both crack surfaces and subtracted, the crack opening displacement $[\![\boldsymbol{u}]\!]$ thus becoming the primary kinematical unknown on the crack.

4 Sensitivity analysis

Consider in the m-dimensional Euclidean space $\mathbf{R}^m$, $m = 2$ or 3, a body Ω_p whose shape depends on a finite number of shape parameters $\boldsymbol{p} = (p_1, p_2, \ldots)$. Shape parameters are treated as time-like parameters using a continuum kinematics-type Lagrangian description and initial configuration Ω_0 conventionally associated with $\boldsymbol{p} = \boldsymbol{0}$ (Petryk and Mroz, 1986):

$$\boldsymbol{x} \in \Omega_0 \rightarrow \boldsymbol{x}^p = \boldsymbol{\Phi}(\boldsymbol{x},\boldsymbol{p}) \in \Omega_p \qquad (8)$$

with $\boldsymbol{\Phi}(\boldsymbol{x},\boldsymbol{0}) = \boldsymbol{x}$ $(\forall \boldsymbol{x} \in \Omega_0)$. The geometrical transformation $\boldsymbol{\Phi}(\cdot,\boldsymbol{p})$ must possess a strictly positive Jacobian for any given $\boldsymbol{p}$. As far as first-order derivatives with respect to $\boldsymbol{p}$ are concerned, attention can be restricted to the consideration of a single shape parameter p without loss of generality.

The *initial transformation velocity* field $\boldsymbol{\theta}(\boldsymbol{x})$, defined by

$$\boldsymbol{\theta}(\boldsymbol{x}) = \frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{p}}(\boldsymbol{x}, p = 0) \qquad (9)$$

is the 'initial' velocity of the 'material' point which coincides with the geometrical point $\boldsymbol{x}$ at 'time' $p = 0$.

The following relations hold between the total (or 'lagrangian', or 'material') derivative $f_p = df/dp$ and the partial (or 'eulerian') derivative $f^p = \partial f/\partial p$ of any sufficiently regular function $f(\boldsymbol{x},p)$:

$$f_p = f^p + \nabla f.\boldsymbol{\theta} \qquad (\nabla f)_p = \nabla(f_p) - \nabla f.\nabla.\boldsymbol{\theta} \qquad (10)$$

The material derivatives of domain and boundary integrals are expressed by (see e.g. Petryk and Mroz, 1986):

$$\frac{d}{dp}\int_{\Omega} f\, \mathrm{d}\Omega = \int_{\Omega} (f_p + f\,\mathrm{div}\boldsymbol{\theta})\, \mathrm{d}\Omega \qquad \Omega\text{: any domain} \qquad (11)$$
$$\frac{d}{dp}\int_{S} f\, \mathrm{d}S = \int_{S} (f_p + f\,\mathrm{div}_S\boldsymbol{\theta})\, \mathrm{d}S \qquad S\text{: any surface} \qquad (12)$$

The surface divergence is given by $\mathrm{div}_S\boldsymbol{\theta} = \mathrm{div}\boldsymbol{\theta} - \boldsymbol{n}.\nabla\boldsymbol{\theta}.\boldsymbol{n}$, where $\boldsymbol{n}$ is the unit normal vector.

One assumes here that the external boundary S and its neighbourhood is unaffected by the shape transformation, so $\boldsymbol{\theta} = \boldsymbol{0}$ and $\nabla\boldsymbol{\theta} = \boldsymbol{0}$ on S. However, this is not true when emerging cracks are considered: in this case, $\boldsymbol{\theta}$ and $\nabla\boldsymbol{\theta}$ do not vanish on some neighbourhood of the emerging point (or edge in 3-D problems).

5 Shape sensitivity: adjoint problem and domain integral formulation

Introduce the following Lagrangian, in which the weak formulation of the direct problem (1, 2) appears as an equality constraint term added to the objective function J:

$$L(\boldsymbol{u},\boldsymbol{v},\Gamma) = J(\boldsymbol{u},\Gamma) + \int_0^T \int_{\Omega} [\boldsymbol{\sigma}(\boldsymbol{u}):\nabla(\boldsymbol{v}) + \rho\ddot{\boldsymbol{u}}.\boldsymbol{v}]\, \mathrm{d}\Omega\, \mathrm{d}t$$
$$- \int_0^T \int_S \bar{\boldsymbol{f}}.\boldsymbol{v}\, \mathrm{d}S\, \mathrm{d}t \qquad (13)$$

Taking into account Eqs. (10)–(12), the total material derivative of the Lagrangian with respect to a variation of the

domain can be expressed as:

$$\frac{d}{dp}L(u,v,\Gamma) = \int_0^T\int_\Omega [\boldsymbol{\sigma}(\boldsymbol{u}_p):\boldsymbol{\nabla}(\boldsymbol{v}) + \rho\ddot{\boldsymbol{u}}_p.\boldsymbol{v}]\,\mathrm{d}\Omega\,\mathrm{d}t + \int_0^T\int_S \frac{\partial\varphi}{\partial\boldsymbol{u}}.\boldsymbol{u}_p\,\mathrm{d}S\,\mathrm{d}t + \int_0^T\int_\Omega [\boldsymbol{\sigma}(\boldsymbol{u}):\boldsymbol{\nabla}(\boldsymbol{v}) + \rho\ddot{\boldsymbol{u}}.\boldsymbol{v}]\,\mathrm{div}\boldsymbol{\theta}\,\mathrm{d}\Omega\,\mathrm{d}t - \int_0^T\int_\Omega [\boldsymbol{\sigma}(\boldsymbol{u}).\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\sigma}(\boldsymbol{v}).\boldsymbol{\nabla}\boldsymbol{u}]:\boldsymbol{\nabla}\boldsymbol{\theta}\,\mathrm{d}\Omega\,\mathrm{d}t \quad (14)$$

For cracks, the partial derivative $(\boldsymbol{\nabla}u)^p$ has generally a $d^{-3/2}$ singularity along the crack edge $\partial\Gamma$, while $\boldsymbol{\nabla}(\boldsymbol{u}_p)$ and $\boldsymbol{\nabla}\boldsymbol{u}$ have the same $d^{-1/2}$ singularity, where d is a distance to $\partial\Gamma$. For this reason, the total derivative $\boldsymbol{u}_p$ has been introduced instead of the partial derivative $\boldsymbol{u}^p$. The derivations made in this section are therefore valid for both void and crack problems.

At this point, it is useful to remark that since the initial conditions $\boldsymbol{u}(\cdot,0) = \dot{\boldsymbol{u}}(\cdot,0)$ hold for any location of the assumed defect, one should assume $\boldsymbol{u}_p(\cdot,0) = \dot{\boldsymbol{u}}_p(\cdot,0)$ as well. One then has:

$$\int_0^T \ddot{\boldsymbol{u}}.\boldsymbol{v}\,\mathrm{d}t = (\dot{\boldsymbol{u}}.\boldsymbol{v} - \dot{\boldsymbol{v}}.\boldsymbol{u})\,|_{t=T} + \int_0^T \boldsymbol{u}.\ddot{\boldsymbol{v}}\,\mathrm{d}t \quad (15)$$

$$\int_0^T \ddot{\boldsymbol{u}}_p.\boldsymbol{v}\,\mathrm{d}t = (\dot{\boldsymbol{u}}_p.\boldsymbol{v} - \dot{\boldsymbol{v}}.\boldsymbol{u}_p)\,|_{t=T} + \int_0^T \boldsymbol{u}_p.\ddot{\boldsymbol{v}}\,\mathrm{d}t \quad (16)$$

In equation (14), the test function $\boldsymbol{v}$ is now chosen so that the terms which contain $\boldsymbol{u}_p$ combine to zero for any $\boldsymbol{u}_p$. Using Eq. (15), one gets:

$$\int_0^T\int_\Omega [\boldsymbol{\sigma}(\boldsymbol{v}):\boldsymbol{\nabla}(\boldsymbol{u}_p) + \rho\ddot{\boldsymbol{v}}.\boldsymbol{u}_p]\,\mathrm{d}\Omega\,\mathrm{d}t + \int_0^T\int_S \frac{\partial\varphi}{\partial\boldsymbol{u}}.\boldsymbol{u}_p\,\mathrm{d}S\,\mathrm{d}t + \int_\Omega (\dot{\boldsymbol{u}}_p.\boldsymbol{v} - \dot{\boldsymbol{v}}.\boldsymbol{u}_p)\,|_{t=T}\,\mathrm{d}\Omega = 0 \qquad (\forall\boldsymbol{u}_p) \quad (17)$$

This last result is the weak formulation, with unknown $\boldsymbol{v}$, of the *adjoint problem*. Thus $\boldsymbol{v}$ solves the field equations (1) together with the following boundary and *final* conditions:

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{v}) &= -\frac{\partial\varphi}{\partial\boldsymbol{u}} && \text{on } S \\ \boldsymbol{f}(\boldsymbol{v}) &= 0 && \text{on } \Gamma \\ \boldsymbol{v} = \dot{\boldsymbol{v}} &= 0 && \text{in } \Omega, \text{ at } t = T \end{aligned} \quad (18)$$

The adjoint problem can be solved in the same way as the direct problem (1, 2), e.g. using the dual reciprocity formulation (eq.(6)), but with time reversed.

Finally, Eq. (14) allows to express the derivative of J in terms of the direct and adjoint solutions:

$$u_\Gamma\frac{d}{dp}J(\Gamma) = \frac{d}{dp}L(u_\Gamma,v_\Gamma,\Gamma) = \int_0^T\int_\Omega [\boldsymbol{\sigma}(\boldsymbol{u}):\boldsymbol{\varepsilon}(\boldsymbol{v}) + \rho\ddot{\boldsymbol{u}}.\boldsymbol{v}]\,\mathrm{div}\boldsymbol{\theta}\,\mathrm{d}\Omega\,\mathrm{d}t - \int_0^T\int_\Omega [\boldsymbol{\sigma}(\boldsymbol{u}).\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\sigma}(\boldsymbol{v}).\boldsymbol{\nabla}\boldsymbol{u}]:\boldsymbol{\nabla}\boldsymbol{\theta}\,\mathrm{d}\Omega\,\mathrm{d}t \quad (19)$$

6 Shape sensitivity: boundary integral formulation (void problem)

The formula (19) for the sensitivity of J is expressed by a domain integral. It is therefore not suitable for BEM-based computations. This section aims to show that Eq. (19) applied to the void problem can be converted into an equivalent, boundary-only, expression.

Besides, it is easy to prove (for example using component notation) that the solutions $\boldsymbol{u}_\Gamma$ and $\boldsymbol{v}_\Gamma$ to the direct and adjoint problems (and, indeed, any pair $(\boldsymbol{u},\boldsymbol{v})$ solving the field equations (1) together with homogeneous initial and final conditions, respectively) verify:

$$\int_0^T \Big\{[\boldsymbol{\sigma}(\boldsymbol{u}):\boldsymbol{\varepsilon}(\boldsymbol{v}) + \rho\ddot{\boldsymbol{u}}.\boldsymbol{v}]\mathrm{div}\boldsymbol{\theta} - [\boldsymbol{\sigma}(\boldsymbol{u}).\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\sigma}(\boldsymbol{v}).\boldsymbol{\nabla}\boldsymbol{u}]:\boldsymbol{\nabla}\boldsymbol{\theta}\Big\}\,\mathrm{d}t = \int_0^T \mathrm{div}\Big([\boldsymbol{\sigma}(\boldsymbol{u}):\boldsymbol{\varepsilon}(\boldsymbol{v}) - \rho\dot{\boldsymbol{u}}.\dot{\boldsymbol{v}}]\boldsymbol{\theta} - [\boldsymbol{\sigma}(\boldsymbol{u}).\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\sigma}(\boldsymbol{v}).\boldsymbol{\nabla}\boldsymbol{u}].\boldsymbol{\theta}\Big)\,\mathrm{d}t \quad (20)$$

(where the subscript Γ in (u_Γ, v_Γ) has been removed for convenience). This identity is then substituted into Eq. (19), leading to a boundary-only expression via the divergence formula:

$$\frac{dJ}{dp} = \int_0^T\int_{\partial\Omega} [\boldsymbol{\sigma}(\boldsymbol{u}):\boldsymbol{\nabla}\boldsymbol{v} - \rho\dot{\boldsymbol{u}}.\dot{\boldsymbol{v}}]\theta_n\,\mathrm{d}S\,\mathrm{d}t - \int_0^T\int_{\partial\Omega} [\boldsymbol{f}(\boldsymbol{u}).\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{f}(\boldsymbol{v}).\boldsymbol{\nabla}\boldsymbol{u}].\boldsymbol{\theta}\,\mathrm{d}S\,\mathrm{d}t \quad (21)$$

provided all integrals involved in the previous steps are convergent (this provision will prove important for crack problems).

Since $\boldsymbol{\theta} = \boldsymbol{0}$ on S and $\boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{f}(\boldsymbol{v}) = \boldsymbol{0}$ on Γ, the above equation reduces to:

$$\frac{dJ}{dp} = \int_0^T\int_\Gamma [\boldsymbol{\sigma}(\boldsymbol{u}):\boldsymbol{\nabla}\boldsymbol{v} - \rho\dot{\boldsymbol{u}}.\dot{\boldsymbol{v}}]\theta_n\,\mathrm{d}S\,\mathrm{d}t \quad (22)$$

The general expression of the bilinear form $\boldsymbol{\sigma}(\boldsymbol{u}):\boldsymbol{\nabla}\boldsymbol{v}$ in terms of $\boldsymbol{\nabla}_S\boldsymbol{u}, \boldsymbol{\nabla}_S\boldsymbol{v}$ and $\boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{f}(\boldsymbol{v})$ (assuming isotropic elasticity and taking $\boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{f}(\boldsymbol{v}) = \boldsymbol{0}$ into account) is:

$$\boldsymbol{\sigma}(\boldsymbol{u}):\boldsymbol{\nabla}\boldsymbol{v} = \mu\Big\{\frac{2\nu}{1-\nu}\mathrm{div}_S\boldsymbol{u}\,\mathrm{div}_S\boldsymbol{v} + \frac{1}{2}(\boldsymbol{\nabla}_S\boldsymbol{u} + \boldsymbol{\nabla}_S^T\boldsymbol{u}):\boldsymbol{\nabla}_S\boldsymbol{v} + \frac{1}{2}(\boldsymbol{\nabla}_S\boldsymbol{v} + \boldsymbol{\nabla}_S^T\boldsymbol{v}):\boldsymbol{\nabla}_S\boldsymbol{u} - (\boldsymbol{n}.\boldsymbol{\nabla}_S\boldsymbol{u}).(\boldsymbol{n}.\boldsymbol{\nabla}_S\boldsymbol{v})\Big\} \quad (23)$$

(ν: Poisson ratio, μ: shear modulus). Its substitution into Eq. (22) produces an expression of $\partial J/\partial p$ in terms of the fields $(\boldsymbol{u},\boldsymbol{v})$ and their tangential derivatives, i.e. well suited to computation.

7 Shape sensitivity: boundary integral formulation (crack problem)

Consider the case where the unknown defect is a crack, i.e. the limiting case of a void bounded by two surfaces Γ^+ and Γ^- identical and of opposite orientations (Fig. 2). It is tempting to still

apply Eq. (21) to compute sensitivities with respect to crack location perturbations. However, Eq. (21) is not correct for crack defects. For instance, consider a domain shape transformation such that $\theta_n = 0$ on the crack surface Γ. This means that crack perturbations along the tangent plane at the crack front are allowed. But then Eq. (21) gives $\mathrm{d}J/\mathrm{d}p = 0$, which is certainty not true in general. In contrast, when Γ is the piecewise smooth boundary of a void, $\theta_n = 0$ implies that the void is unperturbed.

This apparent paradox may be explained as follows: to establish the boundary-only expression (21), one needs an integration by parts using identity (20). On the other hand, Eq. (20) involves the quantity $\mathrm{div}(\boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\nabla v})$, which behaves like d^{-2} in the vicinity of the crack tip (2D) or front (3D) and is therefore not integrable (d: distance to the crack tip or front).

This section aims at showing that these difficulties can be overcome in two instances: (i) consider special cases of domain transformations where the domain integral disappears or is easily transformed, or (ii) additive decomposition of the transformation velocity field $\boldsymbol{\theta}$ in neighbourhoods of the crack tips into a constant and a complementary term.

7.1 Special cases of domain transformations

Isolate a neighbourhood $D \subset \Omega$ of the crack bounded by the surface $\partial D = C$ (Fig. 3) and consider the transformation velocity fields $\boldsymbol{\theta}$ associated with special crack shape transformations: (a) translation of D, (b) expansion of D and (c) rotation of D. These shape transformations are continuously extended so that $\boldsymbol{\theta} = \boldsymbol{0}$, $\boldsymbol{\nabla\theta} = \boldsymbol{0}$ on S. Then, Eq. (21) is valid for the subdomain $\Omega \setminus D$ while advantage is taken of the special form of $\boldsymbol{\theta}$ in D:

(a) translation: $\boldsymbol{\theta} = \boldsymbol{\theta}_0$ (constant) in D, hence $\boldsymbol{\nabla\theta} = \boldsymbol{0}$, $\mathrm{div}\boldsymbol{\theta} = 0$ in D and the domain integral over D in Eq. (19) vanishes;

(b) expansion with respect to the origin: $\boldsymbol{\theta} = \eta\boldsymbol{y}$ (η: expansion coefficient) so that $\boldsymbol{\nabla\theta} = \eta\boldsymbol{I}$, $\mathrm{div}\boldsymbol{\theta} = m\eta$ (m: space dimensionality) in D. In this case, the domain integral over D in Eq. (19) becomes:

$$\eta \int_D \left\{ (m-2)\boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\nabla v} + m\rho\dot{\boldsymbol{u}}.\dot{\boldsymbol{v}} \right\} \mathrm{d}\Omega = \eta(m-2)\int_C \boldsymbol{f}(\boldsymbol{u}).\boldsymbol{v}\,\mathrm{d}S - 2\rho\eta\int_D \dot{\boldsymbol{u}}.\dot{\boldsymbol{v}}\,\mathrm{d}\Omega \quad (24)$$

(c) rotation: $\boldsymbol{\theta} = \boldsymbol{\omega y}$ ($\boldsymbol{\omega}$: constant tensor such that $\boldsymbol{\omega} + \boldsymbol{\omega}^T = \boldsymbol{0}$) so that $\boldsymbol{\nabla\theta} + \boldsymbol{\nabla\theta}^T = \boldsymbol{0}$ and $\mathrm{div}\boldsymbol{\theta} = 0$. Using the identity $\boldsymbol{\nabla w} = 2\boldsymbol{\varepsilon}(\boldsymbol{w}) - \boldsymbol{\nabla}^T\boldsymbol{w}$, the domain integral over D becomes, in component notation:

$$\omega_{aj}\int_D \left\{ \sigma_{ij}(\boldsymbol{u})[v_{a,i} - 2\varepsilon_{ai}(\boldsymbol{v})] + \sigma_{ij}(\boldsymbol{v})[u_{a,i} - 2\varepsilon_{ai}(\boldsymbol{u})] \right\} \mathrm{d}\Omega \quad (25)$$

For isotropic elasticity (λ, μ: Lame constants), one has

$$\sigma_{ij}(\boldsymbol{u})\varepsilon_{ai}(\boldsymbol{v}) + \sigma_{ij}(\boldsymbol{v})\varepsilon_{ai}(\boldsymbol{u}) = \lambda[(\mathrm{div}\boldsymbol{u})\varepsilon_{ja}(\boldsymbol{v}) + (\mathrm{div}\boldsymbol{v})\varepsilon_{ja}(\boldsymbol{u})] + 2\mu[\varepsilon_{ij}(\boldsymbol{u})\varepsilon_{ia}(\boldsymbol{v}) + \varepsilon_{ij}(\boldsymbol{v})\varepsilon_{ia}(\boldsymbol{u})] \quad (26)$$

which is symmetric with respect to the indices (a, j), so that the inner product of this quantity with ω_{aj} vanishes. As a result, the integral over D, Eq. (25), becomes, after application of the divergence formula, integration in time over $[0, T]$ and using initial conditions on $\boldsymbol{u}$ and final conditions on $\boldsymbol{v}$:

$$\omega_{aj}\int_0^T \left\{ \int_C [p_i(\boldsymbol{u})v_a + p_i(\boldsymbol{v})u_a]\,\mathrm{d}S + \int_D [\dot{u}_a\dot{v}_j + \dot{u}_j\dot{v}_a]\,\mathrm{d}\Omega \right\} \mathrm{d}t$$

But the second integral in the above equation is symmetric with respect to the indices (a, j); thus its inner product with ω_{aj} vanishes and only the first term remains.

Collecting all results, we have for cases (a), (b) and (c) together (with $\boldsymbol{\theta} = \boldsymbol{\theta}_0 + \eta\boldsymbol{x} + \boldsymbol{\omega}.\boldsymbol{x}$):

$$\begin{aligned}\frac{\partial J}{\partial p} = &\int_0^T\int_C [\rho\dot{\boldsymbol{u}}.\dot{\boldsymbol{v}} - \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\nabla v}]\theta_n\,\mathrm{d}S\,\mathrm{d}t \\ &+ \int_0^T\int_C [\boldsymbol{f}(\boldsymbol{u}).\boldsymbol{\nabla v} + \boldsymbol{f}(\boldsymbol{v}).\boldsymbol{\nabla u}].\boldsymbol{\theta}\,\mathrm{d}S\,\mathrm{d}t \\ &+ \eta(m-2)\int_0^T\int_C \boldsymbol{f}(\boldsymbol{u}).\boldsymbol{v}\,\mathrm{d}S\,\mathrm{d}t - 2\rho\eta\int_0^T\int_D \dot{\boldsymbol{u}}.\dot{\boldsymbol{v}}\,\mathrm{d}\Omega\,\mathrm{d}t \\ &+ \omega_{aj}\int_0^T\int_C [p_i(\boldsymbol{u})v_a + p_i(\boldsymbol{v})u_a]\,\mathrm{d}S\,\mathrm{d}t \end{aligned} \quad (27)$$

The neighbourhood D of boundary S surrounding the crack is arbitrary. In case (b), due to the presence of the domain integral over D, the sensitivity of the functional J, as expressed by equation (27), is neither a true boundary-only expression, nor true path-independent integral, even if it does not depend on the choice of the surface C.

The special domain transformations considered here follow the idea introduced for elastostatics in Dems and Mróz, 1986 and for time-harmonic problems in Dems and Mróz, 1995, where it

Figure 2. A crack bounded by two almost identical surfaces Γ^+ and Γ^-.

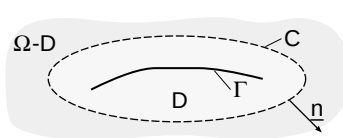

Figure 3. A crack C with a neighbourhood D

is proved that conservation rules and path-independent integrals can be derived for the same special domain transformations. This idea was numerically implemented using boundary elements for sensitivity analysis of cracks (Burczyński and Polch, 1994) and voids (Burczyński and Habarta, 1995) in static problems. This section is then an extension of these previous works to time-domain dynamical problems.

7.2 Additive decomposition of transformation velocity near crack tips

The development to follow is valid for two-dimensional problems only. Introduce neighbourhoods $D_i \subset \Omega$ $(i = 1,2)$ of the two crack tips $\boldsymbol{x}^i$; the boundary of D_i is denoted C_i (Fig. 4). Put $\Gamma_i = \Gamma \cap D_i$, $\bar{\Gamma} = \Gamma \setminus (\Gamma_1 \cup \Gamma_2)$ and $\bar{\Omega} = \Omega \setminus (D_1 \cup D_2)$. Then one has $\partial D_i = C_i \cup \Gamma_i$ and $\partial\Omega = S \cup C_1 \cup C_2 \cup \Gamma$. Let

$$\boldsymbol{\mu} = \boldsymbol{\theta} \quad (\text{in } \bar{\Omega}) \qquad\qquad \boldsymbol{\mu} = \boldsymbol{\theta} - \boldsymbol{\theta}^i \quad (\text{in } D_i,\ i = 1,2) \tag{28}$$

where $\boldsymbol{\theta}^i = \boldsymbol{\theta}(\boldsymbol{x}^i)$ is the transformation velocity at crack tip i.

One has $\boldsymbol{\nabla}\boldsymbol{\mu} = \boldsymbol{\nabla}\boldsymbol{\theta}$ and $\operatorname{div}\boldsymbol{\mu} = \operatorname{div}\boldsymbol{\theta}$ everywhere in each subdomain $\bar{\Omega}, D_1, D_2$. Besides, $\boldsymbol{\mu}(\boldsymbol{x}) = o(1)$ at points $\boldsymbol{x}$ sufficiently close to a tip, so that $\boldsymbol{\nabla}(\boldsymbol{\nabla}u_\Gamma.\boldsymbol{\nabla}v_\Gamma).\boldsymbol{\mu}$ is integrable at crack tips. Then the integration by parts on Eq. (19), carried out separately in each subdomain $\bar{\Omega}, D_1, D_2$ and with $\boldsymbol{\theta}$ replaced by $\boldsymbol{\mu}$, results in Eq. (21) with $\partial\Omega = \partial\bar{\Omega}, \partial D_1, \partial D_2$ respectively.

Adding the contributions for the three subdomains and keeping in mind that $\boldsymbol{\mu}$ is discontinuous across C_i, one obtains:

$$\begin{aligned}\frac{d}{dp}J(\Gamma) = &\int_0^T\int_\Gamma [\![\boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\nabla}\boldsymbol{v} - \rho\dot{\boldsymbol{u}}.\dot{\boldsymbol{v}}]\!]\,\mu_n\,\mathrm{d}S\,\mathrm{d}t \\ &- \sum_{i=1}^{2}\int_0^T\int_{C_i} [\boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\nabla}\boldsymbol{v} - \rho\dot{\boldsymbol{u}}.\dot{\boldsymbol{v}}]\,(\boldsymbol{\theta}^i.\boldsymbol{n})\,\mathrm{d}S\,\mathrm{d}t \\ &+ \sum_{i=1}^{2}\int_0^T\int_{C_i} [\boldsymbol{f}(\boldsymbol{v}).\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{f}(\boldsymbol{u}).\boldsymbol{\nabla}\boldsymbol{v}].\boldsymbol{\theta}^i\,\mathrm{d}S\,\mathrm{d}t\end{aligned} \tag{29}$$

where the various normal vectors are as indicated in Fig. 4. The appearance of $\boldsymbol{\theta}^i$ on C_i stems from the fact that, combining boundary integrals associated with subdomains $\bar{\Omega}$ and D_i, the jump of $\boldsymbol{\mu}$ across C_i appears and is equal to:

$$[\![\boldsymbol{\mu}]\!] = \boldsymbol{\mu}|_{\bar{\Omega}} - \boldsymbol{\mu}\,|_{D_i} = \boldsymbol{\theta} - (\boldsymbol{\theta} - \boldsymbol{\theta}^i) = \boldsymbol{\theta}^i \tag{30}$$

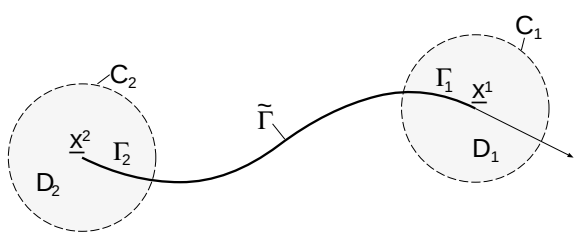

Figure 4. Isolation of crack tips $\boldsymbol{x}^i$ by neighbourhoods D_i $(i = 1,2)$: notations.

The integral on Γ in Eq. (29) is convergent since $\boldsymbol{\mu}$ is built so as to vanish at the crack tips.

The sensitivity expression (29) is general in the sense that it holds for any transformation velocity $\boldsymbol{\theta}$ sufficiently smooth; it is not restricted to simple shape transformations.

The idea of isolating the crack tip and using decomposition (28) is not easily transposable to three-dimensional situations. To discuss this point, let $\partial\Gamma$, a regular closed curve, denote the crack front. Introduce a neighbourhood $D \subset \Omega$ of $\partial\Gamma$ (e.g. of tubular shape) and let $\tilde{\boldsymbol{\theta}}$ denote an arbitrarily chosen extension to D of the restriction of $\boldsymbol{\theta}$ to the front $\partial\Gamma$. The three-dimensional equivalent of Eq. (28) consists of putting

$$\boldsymbol{\mu} = \boldsymbol{\theta} \quad (\text{in } \bar{\Omega}) \qquad\qquad \boldsymbol{\mu} = \boldsymbol{\theta} - \boldsymbol{\theta}^i \quad (\text{in } D) \tag{31}$$

However, due to the curvature of $\partial\Gamma$ and the variability of $\boldsymbol{\theta}$ along $\partial\Gamma$, no choice of the extension is expected to make the domain integral (19) over D vanish. This is at variance with the two-dimensional case, where the constant extension $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta}(\boldsymbol{x}^i)$ is used.

8 Sensitivity formula using stress intensity factors

Boundary element methods are well suited to the evaluation of stress intensity factors (SIFs), e.g. using quarter-node boundary elements that model the square-root local behavior about the crack tips. Application of BEM to numerical evaluation of dynamic SIFs has been investigated e.g. by Fedelinski et al., 1996.

Assume then that the dynamical SIFs $K_I^u(t;\boldsymbol{x}^i), K_{II}^u(t;\boldsymbol{x}^i)$, $K_I^v(t;\boldsymbol{x}^i), K_{II}^v(t;\boldsymbol{x}^i)$ at tip $\boldsymbol{x}^i$, associated with the solutions of the primary and adjoint problems, respectively, are known. Since the curves C_i are arbitrary, one may follow the procedure that allows to link J-integral to SIFs, i.e. assume that C_i is the circle of radius ε centered at crack tip $\boldsymbol{x}^i$ and investigate the limiting case when $\varepsilon \to 0$. In the vicinity of a (non-moving) crack tip, the following, well-known, expansions hold:

$$\begin{aligned}w_r = \frac{1}{2\mu}\sqrt{\frac{d}{2\pi}}\Big[&K_I^w(t)\cos\frac{\theta}{2}(3 - 4\nu - \cos\theta) \\ &+ K_{II}^w(t)\sin\frac{\theta}{2}(4\nu - 1 + 3\cos\theta)\Big] + O(d)\end{aligned} \tag{32}$$

$$\begin{aligned}w_\theta = \frac{1}{2\mu}\sqrt{\frac{d}{2\pi}}\Big[&-K_I^w(t)\sin\frac{\theta}{2}(1 - 4\nu - 3\cos\theta) \\ &+ K_{II}^w(t)\cos\frac{\theta}{2}(4\nu - 5 + 3\cos\theta)\Big] + O(d)\end{aligned} \tag{33}$$

where $\boldsymbol{w}$ is replaced by either $\boldsymbol{u}$ or $\boldsymbol{v}$; (ρ,θ) denote polar coordinates emanating from the crack tip as indicated on Fig. 5.

First, one notes that the strain energy density $\boldsymbol{\sigma}(\boldsymbol{w}) : \boldsymbol{\varepsilon}(\boldsymbol{w})$ calculated from expansions (32–33) is d^{-1}-singular but continuous across the crack Γ. This remark implies that:

$$\lim_{\varepsilon\to 0}\int_{\Gamma_i} [\![\boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\nabla}\boldsymbol{v} - \rho\dot{\boldsymbol{u}}.\dot{\boldsymbol{v}}]\!]\,\mu_n\,\mathrm{d}S = 0$$

$$\lim_{\varepsilon\to 0}\int_{\bar{\Gamma}}[\![\boldsymbol{\sigma}(\boldsymbol{u}):\boldsymbol{\nabla v}-\rho\dot{\boldsymbol{u}}.\dot{\boldsymbol{v}}]\!]\mu_n\,\mathrm{d}S=\int_{\Gamma}[\![\boldsymbol{\sigma}(\boldsymbol{u}):\boldsymbol{\nabla v}-\rho\dot{\boldsymbol{u}}.\dot{\boldsymbol{v}}]\!]\mu_n\,\mathrm{d}S$$

and also that the resulting line integral over Γ is nonsingular.

Besides, the limiting process $\varepsilon\to 0$ in the line integrals over C_i is carried out.In the limit $\varepsilon\to 0$, Eq. (29) becomes:

$$\begin{aligned}\frac{d}{dp}J(\Gamma)&=\int_0^T\int_{\Gamma}[\![\boldsymbol{\sigma}(\boldsymbol{u}):\boldsymbol{\nabla v}-\rho\dot{\boldsymbol{u}}.\dot{\boldsymbol{v}}]\!]\theta_n\,\mathrm{d}S\,\mathrm{d}t\\&-\frac{1-\nu}{\mu}\sum_{i=1}^{2}\int_0^T\Big[(K_I^u(t;\boldsymbol{x}^i)K_I^v(t;\boldsymbol{x}^i)+K_{II}^u(t;\boldsymbol{x}^i)K_{II}^v(t;\boldsymbol{x}^i))(\theta_\tau^i)\\&\qquad-(K_I^u(t;\boldsymbol{x}^i)K_{II}^v(t;\boldsymbol{x}^i)+K_{II}^u(t;\boldsymbol{x}^i)K_I^v(t;\boldsymbol{x}^i))(\theta_n^i)\Big]\,\mathrm{d}t\end{aligned}\tag{34}$$

where θ_τ^i,θ_n^i are the tangent and normal components (according to notations of Fig. 5), respectively, of the crack-tip velocity. Equation (34) thus provides a convenient means to evaluate the sensitivity of $J(\Gamma)$.

9 Numerical example

A square plate of length $2b=20\,\mathrm{mm}$ contains a central crack of length $2a=10\,\mathrm{mm}$, as shown in Fig. 6. The material properties are: Young modulus $E=2\,10^{11}\,\mathrm{Pa}$, Poisson ratio $\nu=0.3$, mass density $\rho=5000\,\mathrm{kg/m^3}$. The plate is in a state of plane strain. One edge of the plate is constrained, while the opposite edge is loaded by a uniform tensile traction of constant magnitude $q_p=2\,10^8\,\mathrm{N/m}$ applied during the time interval $0\le t\le T=80\,\mu\mathrm{s}$. The boundary is divided into 44 three-noded boundary elements, and 62 additional points are used for the DRM. The time step used for the time discretization is $\Delta t=0.2\,\mu\mathrm{s}$.

In the first example, crack translations along the x_1-direction are considered: the transformation is

$$\boldsymbol{\Phi}(\boldsymbol{x},p)=\boldsymbol{x}+ap\boldsymbol{e}_1$$

and the objective function $J(\Gamma)$ is chosen as:

$$J(\Gamma)=\int_0^T u_{1\Gamma}(M,t)\,\mathrm{d}t\tag{35}$$

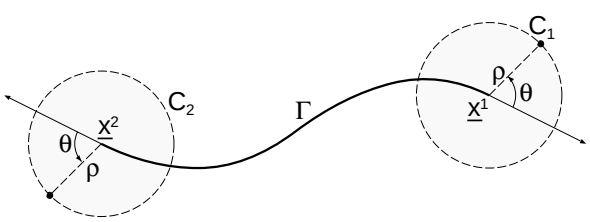

Figure 5. Local polar coordinates and tangent and normal vectors associated with the crack tips.

where the measurement point M is located on the lower side AB of the square plate. Equation (34) yields the result:

$$\frac{d}{dp}J\approx -7.443$$

whereas a second-degree polynomial approximation of J computed from numerical values of J with $-0.15\le p\le 0.15$ yields

$$\frac{d}{dp}J_{polynomial}\approx -6.965$$

and central finite-difference evaluations of $\mathrm{d}J/\mathrm{d}p$ give:

p	$\mathrm{d}J/\mathrm{d}p$
0.01	-5.671
0.02	-6.967
0.04	-6.969
0.08	-6.971
0.10	-6.976

In the second example, the straight crack is deformed into a parabolic shape according to the transformation:

$$\boldsymbol{\Phi}(\boldsymbol{x},p)=\boldsymbol{x}+p(a^2-x_2^2)\boldsymbol{e}_1$$

and the objective function $J(\Gamma)$ is chosen as:

$$-\frac{1}{2}\int_0^T\int_{S_m}u_1^2(\boldsymbol{y},t)\,\mathrm{d}S\,\mathrm{d}t\tag{36}$$

where the measurement surface S_m is the entire lower side AB of the square plate. Equation (34) yields the result:

$$\frac{d}{dp}J\approx 0.490$$

whereas a second-degree polynomial approximation of J computed from numerical values of J with $-0.15\le p\le 0.15$ yields

$$\frac{d}{dp}J_{polynomial}\approx 0.507$$

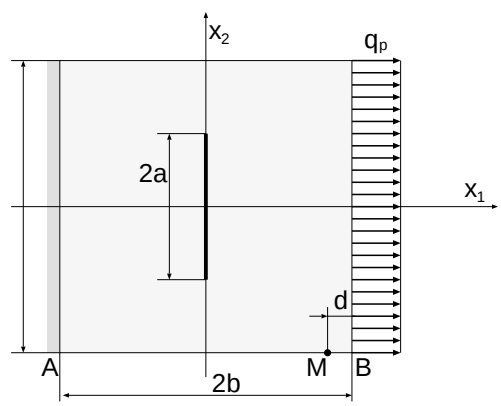

Figure 6. Square plate: geometry and notations.

and central finite-difference evaluations of $\mathrm{d}J/\mathrm{d}p$ give:

p	$\mathrm{d}J/\mathrm{d}p$
0.002	0.507
0.004	0.507
0.01	0.507
0.04	0.507

One notices that the agreement between the sensitivity formula (34) and finite-difference evaluations is better for the second example. This may be attributable to the fact that in the first example the adjoint problem is defined in terms of a point force (applied at M), which is not well approximated using the usual boundary element interpolation of tractions, whereas the adjoint traction for the second example is distributed over a measurement surface.

The above numerical experiments are the first we attempted to test this particular strategy for computing crack shape sensitivities. More complete numerical tests are in progress. The inclusion of this approach in an strategy using a gradient-based minimization algorithm is the next step of this study.

10 Concluding remarks

In the present work a shape sensitivity analysis for identification of internal defects such as voids and cracks has been presented. The main motivation of this paper was to explore the adjoint variable approach, in the presence of cracks and in connexion with BIE formulations of the direct problem.

A general formulation for the sensitivity of objective functional expressing a distance between given (measured) and computed (for an assumed defect) values of the supplementary boundary data with respect to shape and position of a void has been derived using the material derivative-adjoint variable approach. The sensitivity of the functional has been expressed as a boundary integral.

In the case of a crack, the previous boundary-only expression is not applicable. However, revisiting the discussion of the void problem, the adjoint variable approach has been shown to be still applicable to sensitivity analysis in the presence of cracks for two classes of crack perturbations. Firstly, when the domain transformations considered consist of crack translations, rotations or expansions, the functional sensitivity is expressed as an integral over an arbitrary surface surrounding the crack, supplemented for the case of crack expansion in dynamics by a domain integral over the crack front neighbourhood enclosed by this surface. This applies for arbitrary geometries, either three- and two-dimensional. Earlier works on path-independent integral approach to sensitivity analysis are thus revisited and generalized. Secondly, new sensitivity formulas have been established for general two-dimensional geometries and crack perturbations; they involve only line integrals on the crack curve and on arbitrary closed curves isolating the crack tips.

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