ME05

A FIXED POINT ALGORITHM FOR THE INVERSE SOLUTION OF FLUID FLOW EQUATIONS

Luis Carlos de Castro SANTOS

Departamento de Matemática Aplicada, Universidade de São Paulo Rua do Matão, 1010 São Paulo,SP Brazil Isantos@ime.usp.br

Keywords: Inverse design, Garabedian McFadden method, Fixed point iteration

ABSTRACT

The development of several fluid mechanics applications lead to inverse problems. Given a required distribution of flow variables one has to find the corresponding geometry which provides such distribution. Since the flow governing equations do not allow explicit inversion, iterative methods are used. A linear auxiliary equation, which is a simplified model of the flow governing equations, can be used to develop a fixed point iterative method. Such equation is used to compute the geometrical correction required to minimize the difference between the required and actual flow variable distributions. The auxiliary inverse problem is coupled to a flow solver (from potential flow to Navier-Stokes, the method is solver independent) to iterate the correction until convergence. An auxiliary equation method, the Modified Garabedian Mc-Fadden, is analyzed. It involves a certain number of arbitrary parameters whose choice affects the rate of convergence. The present work describes how to find adequate parameters for different families of airfoils in transonic flow. A 2D Euler/Navier-Stokes flow solver will be used as an analysis tool and a series of studies, which demonstrate the accuracy and robustness of the technique, are presented.

NOMENCLATURE

A, B, C, D	- coefficients of the auxiliary equation
Δy	- changes in vertical coordinate
Q	- magnitude of velocity
q	- state vector of conserved properties
E, F	- inviscid fluxes
R, S	- viscous fluxes
Re	- Reynolds Number

ρ	- fluid density
u,v	- cartesian components of velocity
e	- total energy
Т	- temperature
р	- pressure
C _v	specific heat at constant volume
$\xi_x, \xi_y, \eta_x, \eta$	_y - metrics terms
J	- Jacobian of the transformation
U, V	- contravarient components of velocity
τ	- viscous stress
$\Phi_{ m R}$, $\Phi_{ m S}$	- viscous dissipation terms
Δt	- time step
A , B	- Jacobian matrices of the fluxes
δ	- difference operator
D_I , D_E	- implicit and explicit dissipation
ϵ_{I} , ϵ_{E}	- dissipation coefficients
M_{∞}	- freestream Mach number
C, R	- residual

INTRODUCTION

Computational Fluid Dynamics (CFD) has been proven an important analysis tool in a vast amount of applications: from aircraft design to weather prediction. Its ability to handle realistic flow conditions is an enormous advantage when compared to other methods. On the other hand the computational cost associated to the solution of the Navier-Stokes equations still severely limits its application to design. This issue has received the a lot of attention from researchers over the years, especially in aerospace engineering⁽¹⁻²⁾. Historically, a successful technique in airfoil design is the inverse approach. Based on designer's experience to judge appropriate pressure distributions for typical flow conditions, numerical methods are used to find out the shape which, under the same flow conditions, would provide the prescribed pressure distribution. The problem is ill-posed since there is no guarantee that the prescribed pressure distribution is feasible. In practice designers start from an initial geometry, analyze it, numerical or experimentally, to obtain the pressure distribution, and then propose some modification.

This paper is concerned on showing how a purely geometrical approach to inverse design methods can be used quite efficiently to airfoil design. Linear auxiliary equations, based on compressible potential flow, can be used to construct a fixed point algorithm. The flow solver computes the pressure distribution which is compared to a target distribution. The residual is an input to an inverse auxiliary equation which provides the correction on the geometry. The profile is corrected and the procedure is repeated until convergence is reached. The Modified Garabedian Mc-Fadden method^(3.4) will be used, to illustrate the main aspects of the auxiliary equation technique, although the whole procedure can be extended to any type of auxiliary equation.

An important issue is the choice of the control parameters necessary to guarantee stability of the method. They should allow fast convergence while maintaining stability in a wide range of flow conditions. A short study will be presented to provide the guidelines in the followed in the prescription of such parameters. Using a least squares method for a family of airfoils some guidelines for the relative values of each arbitrary parameter were obtained⁽⁶⁾. As the design cycles converge the changes on the airfoil geometry are naturally reduced therefore the arbitrary parameters should changed to compensate. Some strategies are proposed and design test cases will be performed..

THE AUXILIARY EQUATION TECHNIQUE

The geometrical inverse design methods are based on residual correction, in which the residuals are the difference between the desired pressure distribution (or equivalent velocity distribution) and the computed distribution.

The Modified Garabedian Mc-Fadden method⁽³⁻⁴⁾ is a method of this type where an auxiliary equation is solved in order to compute the correction in the geometry during each design cycle. For an airfoil the MGM auxiliary equation is given by:

$$A \Delta y + B \Delta y_x + C \Delta y_{xx} = Q^2 + DQ_x^2$$
(1)

Where A, B, C, and D are constants chosen to produce a stable iterative process, and the residual:

$$Q^{2} = Q_{t}^{2} - Q_{c}^{2}$$
(2)

is the difference between the target (Q_t) and actual (Q_c) equivalent velocity distributions. As the residual decreases the initial shape converges to the target.

The auxiliary equation combines results from compressible linear theory, that is, the pressure (or equivalent velocity) is dependent on the profile thickness, local slope and curvature. Since this relation is not explicit the constants are introduced. These constants become a set of arbitrary control parameters. The MGM method uses four parameters controlling thickness, slope, curvature and velocity slope respectively. Other approaches like the DISC method⁽⁸⁻⁹⁾ follow a different approach but also presents a pair of parameters for the geometry correction equation, and an extra one providing the blend from a subsonic to a transonic formula.

Despite its efficiency in inverse design the rate of convergence of the process can be strongly affected by the choice of the coefficients. If the coefficients are set too high the changes on the shape (Δy) can be too small and the rate of convergence will decrease. In opposition if the coefficients are too small large changes in shape may results in unfeasible profiles requiring the addition of some geometry constraints. Another issue is the balance between the change in the coordinate y, its slope and curvature which is far from intuitive.

THE NAVIER-STOKES FLOW SOLVER

The flow phenomena of interest is modeled by the Reynolds-averaged Navier-Stokes equations, here presented in a two-dimensional body-fitted coordinate system:

$$\partial_t q + \partial_\xi E + \partial_\eta F = 1/\text{Re}\left(\partial_\xi R + \partial_\eta S\right) \tag{3}$$

where q is vector of conserved quantities: mass, momentum in x and y directions and energy.

$$q = 1/J \{ \rho, \rho u, \rho v, \rho e \}^{T}$$
(4)

The energy is given by:

$$e = \rho \left[C_v T + (u^2 + v^2)/2 \right]$$
 (5)

where T is the temperature and C_{ν} the specific heat at constant volume.

The vectors E and F are the inviscid fluxes of the conserved quantities in ξ and η directions:

$$E = 1/J \{ \rho U, \rho u U + p \xi_x, \rho v U + p \xi_y, U(e+p) \}^T$$
$$F = 1/J \{ \rho V, \rho u V + p \eta_x, \rho v V + p \eta_y, V(e+p) \}^T$$

(6)

where U and V are the contravarient components of velocity:

$$U=u\xi_{x}+v\xi_{y} , V=u\eta_{x}+v\eta_{y}$$
 (7)

The vectors R and S are the viscous fluxes, respectively in ξ and η directions:

$$\begin{split} R &= 1/J \; \{0, \, \tau_{x\xi} \; , \, \tau_{y\xi} \; 0, \; \Phi_R \; \xi_{\,x} + \; \Phi_S \; \xi_{\,y} \; \} \;^T \\ S &= 1/J \; \{0, \, \tau_{x\eta} \; \; , \; \tau_{y\eta} \; , \; \Phi_R \; \eta \;_x + \; \Phi_S \; \eta \;_y \; \} \;^T \end{split}$$

(8)

and the stresses:

$$\begin{split} \tau_{x\xi} &= \tau_{xx} \, \xi_x + \tau_{xy} \, \xi_y \\ \tau_{y\xi} &= \tau_{xy} \, \xi_x + \tau_{yy} \, \xi_y \\ \tau_{x\eta} &= \tau_{xx} \, \eta_x + \tau_{xy} \, \eta_y \\ \tau_{y\eta} &= \tau_{xy} \, \eta_x + \tau_{yy} \, \eta_y \end{split}$$

with the fluid being considered as Newtonian and Φ_R and Φ_S given as viscous dissipation terms⁽⁷⁾.

The equations are numerically approximated by central differences resulting in:

$$(I + \Delta t \ J\delta_{\xi} \mathbf{A}^{n} + \epsilon_{I} \mathbf{D}_{I\epsilon}) (I + \Delta t \ J\delta_{\eta} \mathbf{B}^{n} + \epsilon_{I} \mathbf{D}_{I\eta}) \{ \ \Delta q \} = \{ \mathbf{C}^{n} - \epsilon_{E} \mathbf{D}_{E} \}$$
(9)

where $\mathbf{A} = \partial E / \partial q$ and $\mathbf{B} = \partial F / \partial q$ are the Jacobian matrices and the residual:

$$\mathbf{C}^{n} = -\Delta t J \left(\delta_{\xi} \mathbf{E} + \delta_{\eta} \mathbf{F} \right) + \Delta t J / \operatorname{Re}(\delta_{\xi} \mathbf{R} + \delta_{\eta} \mathbf{S})$$
(10)

The terms D_I and D_E are artificial dissipation terms⁽¹⁰⁾ required to stabilize the numerical scheme. In addition to that, the algebraic Baldwin-Lomax model is used, to take into account turbulence effects. The problem is de-coupled in two pentadiagonal problems and the computational code developed by Sankar Huff and Wu⁽⁵⁾ is used to obtain the solution. The domain is discretized as shows Figure 1.

LEAST SQUARES ANALYSIS

In order to propose adequate choices of the control parameters, a numerical study is proposed. A certain number of airfoils is selected and a database of solutions is constructed using the 2-D Navier-Stokes flow solver⁽⁵⁾ for a fixed Mach number, angle of attack and Reynolds number.

For the Modified Garabedian Mc-Fadden method, a linear least squares problem can be constructed, by minimizing a residual R for the n points which discretize each profile:

$$R = \prod_{i=1}^{n} (A\Delta y + B\Delta y_{x} + C\Delta y_{xx} - Q^{2} - DQ_{x}^{2})$$
(11)

Requiring the residual to be a minimum, we can write a system of equations for A, B, C and D:

$$[M]{A, B, C, D}^{T} = {b}^{T}$$
(12)

The elements of the matrix [M] are given by:

$$\begin{split} M_{11} &= \Delta y^{2} , & M_{12} &= \Delta y \Delta y_{x} \\ M_{12} &= \Delta y \Delta y_{x} , & M_{13} &= \Delta y \Delta y_{xx} \\ M_{14} &= -\Delta y Q_{x}^{2} , & b_{1} &= -\Delta y Q^{2} \\ M_{22} &= \Delta y_{x}^{2} , & M_{23} &= \Delta y_{x} \Delta y_{xx} \\ M_{24} &= -\Delta y_{x} Q_{x}^{2} , & b_{2} &= -\Delta y_{x} Q^{2} \\ M_{33} &= \Delta y_{xx}^{2} , & M_{34} &= -\Delta y_{xx} Q_{x}^{2} \\ b_{3} &= -\Delta y_{xx} Q^{2} , & b_{3} &= -\Delta y_{xx} Q^{2} \\ M_{44} &= -Q_{x}^{4} , & b_{4} &= -Q_{x}^{2} Q^{2} \\ M_{21} &= M_{12} , M_{31} &= M_{13} , M_{41} &= -M_{14} \\ M_{32} &= M_{23} , M_{42} &= M_{24} , M_{43} &= -M_{34} \end{split}$$



Figure 1 Body-Fitted Grid System for Navier-Stokes calculations

The solution of this least squares problem, for each pair of profiles in the data base provides information on the range of the parameters with respect to shape⁽⁶⁾. Also the trends with respect to the flow condition (M, α , Re) can be evaluated.

NUMERICAL STUDIES

Fixed Settings

To compute the arbitrary parameters of the auxiliary equation 4 profiles were selected: RAE 2822 (p1), NACA 0012

(p2), ONERA M-6 (p3) and NACA 64A210 (p4). To provide the necessary velocity distributions for the least squares problem all the profiles were analyzed for the flow condition: $M_{\infty} = 0.676$, $\alpha = 2.40^{\circ}$, Re = 5,700,000



Figure 2a. RAE2822 Geometry and Pressure Distribution for $M=0.676,\,\alpha=2.40^{\circ}$, Re=5,700,000



Figure 2b. NACA0012 Geometry and Pressure Distribution for M = 0.676, α = 2.40° , Re = 5,700,000

A 247 x 50 grid was used and all computations where converged until the residual (eq. 10) reached 10^{-7} . Each solution is obtained after approximately 30 minutes of CPU time, in a SUN UltraSparc machine, which in terms of turnaround time is quite reasonable. Geometries and Pressure Distributions, for each profile, are presented on figures 2a to 2d respectively. Once the data was collected the least square problem could solved for each pair of profiles, the results can be seen on Tables I to III. The Tables are symmetric, at least to the second digit, which indicates the accuracy of the study. For simplicity the term on $Q_{\rm x}$ was neglected.

One can observe also the clearly distinct value for each of the coefficients, in all cases the parameter which controls the change in coordinate y is on the unity level, the parameter which control the slope in the 0.1 level and the parameter which control the curvature in 0.0001 level. Based on that another study is conducted.



Figure 2c. ONERA M6 Geometry and Pressure Distribution for M = 0.676, α = 2.40°, Re = 5,700,000



Figure 2d. NACA 64A210 Geometry and Pressure Distribution for $M=0.676,\,\alpha=2.40^\circ\,,\,Re=5,700,000$

	p1	p2	ր3	p4
p1	-	2.12888	1.00077	2.60192
p2	2.12669	-	0.65828	2.54662
р3	0.99262	0.65939	-	1.45380
p4	2.60127	2.55319	1.45174	-

Table I : Least Squares res	ults for parameter A
-----------------------------	----------------------

	p1	p2	p3	p4
p1	-	0.20309	0.21992	0.02517
p2	0.20248	-	0.06404	0.17031
p3	0.02110	0.06411	-	0.01855
n4	0.02517	0.17032	0.01850	-

Table II : Least Squares results for parameter B

	p1	p2	p3	p4
p1	-	0.00062	0.00010	0.00010
p2	0.00062	-	0.00038	0.00051
p3	0.00010	0.00038	-	0.00031
p4	0.00010	0.00051	0.00031	-

Table III : Least Squares results for parameter C



Figure 3a. Design History - Constant Parameters A=B=C=1; M = $0.676, \alpha = 2.40^{\circ}$, Re = 5,700,000

As a basis of comparison the first line and column were proposed as design problems. That is the RAE2822 is chosen as an initial geometry and the corresponding solution, for the other three profiles, as target distributions. The reverse was also tested.

Initially the control parameters A = B = C were set to unity and the results can seen in Figures 3a and 3b. One can observe a slow decreased in the average ΔCp after 20 design cycles. On Figure 3a the RAE2822 was used as an initial profile, on Figure 3b the pressure distribution for RAE2822 is given as a target, the design process proved to be fully reversible, although still not very accurate below a certain tolerance.



Figure 3b. Design History - A = B = C = 1; M = 0.676, α = 2.40° , Re = 5,700,000



Figure 4a. Design History - A = 1, B = C = 0.1 ; M = 0.676, α = 2.40°, Re = 5,700,000

A first trial was made using A = 1, B = 0.1 and C = 0.0001, as indicated by the previous least square study. This choice proved to lead to instability due to extremely large Δy . That is mostly due to the influence of curvature. Therefore the parameter C was increased to 0.1. Added to that a maximum 1% change in thickness was allowed in order to limit the change of shape. The design test cases were re-computed and the rate of convergence compared. The results are found on Figures 4a and 4b. One can notice now a much higher rate of convergence due to the better capture of curvature effects. The average level has dropped to a half than on the constant parameter cases.



Figure 4b. Design History - A = 1, B = C = 0.1 ; M = 0.676, α = 2.40° , Re = 5,700,000

For all design studies the grid was reduced to 157 X 40 and in the intermediate design steps, since the changes between each inverse cycle are small, the solution was restarted in order to reduce the overall computational time. Each design run (20 cycles) took an average 40 minutes of CPU in a 6 processors SUN Ultra Sparc machine.

Figures 5a to 5b present corresponding geometries of Figures 4a, the initial profile is the RAE2822 and corresponding pressure distributions of the other three profiles are set as target.



Figure 5a. Initial Profile RAE2822; Dashed Line: Designed Profile; Solid Line: NACA0012; Inverse Design After 20 cycles; $A = 1, B = C = 0.1; M = 0.676, \alpha = 2.40^{\circ}, Re = 5,700,000$



Figure 5b. Initial Profile RAE2822; Dashed Line: Designed Profile; Solid Line: ONERA M6; Inverse Design After 20 cycles; A = 1, B = C = 0.1; M = 0.676, α = 2.40°, Re = 5,700,000



Figure 5c. Initial Profile RAE2822; Dashed Line: Designed Profile; Solid Line: NACA 64A210; Inverse Design After 20 cycles; A = 1, B = C = 0.1; M = 0.676, $\alpha = 2.40^{\circ}$, Re = 5,700,000



Figure 6a. Initial Profile NACA0012; Dashed Line: Designed Profile; Solid Line: RAE2822; Inverse Design After 20 cycles; A = 1, B = C = 0.1; M = 0.676, $\alpha = 2.40^{\circ}$, Re = 5,700,000



Figure 6b. Initial Profile ONERA M6; Dashed Line: Designed Profile; Solid Line: RAE2822; Inverse Design After 20 cycles; A = 1, B = C = 0.1; M = 0.676, $\alpha = 2.40^{\circ}$, Re = 5,700,000

Figures 6a to 6b present corresponding geometries of Figures 4b, the target pressure distribution corresponds to the solution for the RAE2822 and the other three profiles are used as initial geometries. As one can see, in both cases, the match is accurate.



Figure 6c. Initial Profile NACA 64A210; Dashed Line: Designed Profile; Solid Line: RAE2822; Inverse Design After 20 cycles; A = 1, B = C = 0.1; M = 0.676, $\alpha = 2.40^{\circ}$, Re = 5,700,000

Dynamical Adjustment

To study the impact on convergence and accuracy some alternatives will be studied⁽⁷⁾. In all experiments the conditions were similar: A = 1, B = C = 0.1; M = 0.676, $\alpha = 2.40^{\circ}$, Re = 5,700,000. The initial profile is the NACA 0012 and the target pressure distribution corresponds to the pressure distribution for the NACA 64A210 at the same flow conditions. Therefore the design geometry will be compared to the NACA 64A210.

The first experiment was to vary the limitation on the maximum change Δy to verify its impact on convergence. At each design step the maximum change is limited to a certain value (0.5%, 1% and 2% in the experiments) and all the changes are re-scaled to preserve the overall shape. Figure 7 shows that the effect is minimal, which leads us to believe that the application of this constraint does not interfere in convergence or accuracy.



Figure 7. Design History - Constraint in ΔY , solid line (target) , dashed (r=20%), dash-dot (r=50%), dotted (r=95%)

Figures from 8 to 10 show that the only parameter whose influence is noticeable is C, which is related to curvature. On figure 10a the variation of the parameter C along the chord provided some capture of the profile curvature along the whole chord. On figure 10b it is verified that the decrease of the parameter C in successive design cycles introduces undesirable fluctuations.



Figure 8a. Design History - Variation of A along the chord, solid line (target), dashed (r=20%), dash-dot (r=50%), dotted (r=95%)



Figure 8b. Design History - Variation of A along the design cycles, solid line (target), dashed (r=20%), dash-dot (r=50%), dotted (r=95%)

The second set of tests was to vary each of the control parameters independently. For each one the value was reduced first as the design cycles progress, to increase the value of the changes in Δy along the design iterations (a), and second by a parabolic distribution along to chord, in order to increase the changes Δy at the edge where the method is less accurate (b). An additional parameter *r* was used to control the rate of decay in each design cycle, in case (a) and the rate of decay along the chord, in case (b).



Figure 9a. Design History - Variation of B along the chord, solid line (target), dashed (r=20%), dash-dot (r=50%), dotted (r=95%)



Figure 9b. Design History - Variation of B along the design cycles, solid line (target), dashed (r=20%), dash-dot (r=50%), dotted (r=95%)



Figure 10a. Design History - Variation of C along the chord, solid line (target), dashed (r=50%), dash-dot (r=95%), dotted (r=99%)



Figure 10b. Design History - Variation of C along the design cycles, solid line (target) , dashed (r=50%), dash-dot (r=95%), dotted (r=99%)

CONCLUDING REMARKS

This study indicates how the choice of the arbitrary parameters necessary for auxiliary equation inverse design methods can affect the rate of convergence of the design. The modeling of curvature is of utmost important but due to its high sensitivity it requires additional damping (a higher parameter) coupled to a constraint in the maximum Δy . An additional set of experiments lead to the conclusion that the only parameter in which some dynamic adjustment can have an effect on the convergence is the one related to curvature. The other parameters are somewhat inert. Based on that, research will be conducted in order to develop a new algorithm with a better curvature capture, especially in the leading edge region.

REFERENCES

- Vanderplaats, G. N., 1979, "Efficient Algorithm for Numerical Airfoil Optimization," Journal of Aircraft, Vol. 16, no. 12, pp.842-847.
- Giles, M. B. and Drela, M., 1987, "Two-Dimensional Transonic Aerodynamic Design Method," AIAA Journal, Vol. 25, no. 9, pp. 1199-1205,.

- Malone, J. B., Narramore, J. C. and Sankar, L. N., 1989, " An Efficient Airfoil Design Method using the Navier-Stokes Equations," AGARD CP 463, pp. 5.1-5.18,.
- 4. Malone, J. B., Narramore, J. C. and Sankar, L. N., 1991, "Airfoil Design Method using the Navier-Stokes Equations," Journal of The Aircraft, Vol. 28, no. 3, pp. 216-224.
- Huff, D.L., Wu, J., Sankar, L. N., 1987, "Analysis of Viscous Transonic Flow over Airfoil Sections", AIAA paper 87-0420, 25th Aerospace Sciences Meeting, Reno, NV, USA,.
- Santos, L.C. de C., 1998, "Modelling Auxiliary Geometric Equations for Inverse Airfoil Design Methods", AIAA paper 98-2405, 16th AIAA Applied Aerodynamics Conference, Albuquerque, NM, USA.
- Santos, L.C. de C., 1999, "Modelling Auxiliary Geometric Equations for Inverse Airfoil Design Methods", AIAA paper 99-0184, 16th AIAA Meeting and Exhibit, Reno, NV, USA.
- Campbell, R. L. and Smith, L. A.; "A Hybrid Algorithm for Transonic Airofil and Wing Design", Proceedings of the AIAA 5th Applied Aerodynamics Conference, pp. 527-538, 1987.
- Smith, L. A. and Campbell, R. L. ; "A Method for the Design of Transonic Flexible Wings", NASA TP-3045, 1991.
- Pulliam, T. H., 1985, "Artificial Dissipation Models for the Euler Equations", AIAA paper 85-0438, 23th Aerospace Sciences Meeting, Reno, NV, USA.