# AN INVERSE PROBLEM FOR SLOW VISCOUS INCOMPRESSIBLE FLOWS 

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#### Abstract

This paper considers an inverse boundary value problem associated to the Stokes equations which govern the motion of slow viscous incompressible flows of fluids. The determination of the under-specified boundary values of the normal fluid velocity is made possible by utilising within the analysis additional pressure measurements which are available from elsewhere on the boundary. The inverse boundary value Stokes problem has been numerically discretised using a boundary element method (BEM). Then the resulting ill-conditioned system of linear algebraic equations solved in a circular domain using the Tikhonov regularization method with the choice of the regularization parameter based on the $L$-curve criterion. In addition, an investigation into the stability of the numerical solution has been made by adding a random small perturbation to the input data.


## NOMENCLATURE

$A, B, C, D, E, F, G, H \quad$ Influence matrices
$I$ Identity matrix
$K_{k l}, L_{k} \quad$ Kernel functions
$M$ Number of boundary elements
$R$ Radius of the circle
$f_{1}, f_{2}, \bar{f}_{1}, \bar{f}_{2}$ Stress force components
$\underline{n}$ Outward normal
$p, \bar{p} \quad$ Pressure
$(r, \theta)$ Polar coordinates
$u_{k}^{l}, q^{k} \quad$ Fundamental solutions
$\underline{v}$ Fluid velocity vector
$v_{1}, v_{2}, \bar{v}_{1}, \bar{v}_{2}, v_{r}, v_{\theta}$ Components of the fluid velocity
$w$ Fluid vorticity
$\left(x_{1}, x_{2}\right),\left(\bar{x}_{1}, \bar{x}_{2}\right) \quad$ Cartesian coordinates
$\Gamma_{0}, \Gamma^{\star}$ Under- and over-specified portions of the boundary, respectively
$\Omega$ Solution domain
$\alpha$ Percentage of noise
$\delta$ Kronecker delta symbol
$\epsilon$ Gaussian random variables
$\eta$ Coefficient function
$\phi, \chi, \xi$ Given functions
$\lambda$ Regularization parameter
$\psi$ Streamfunction
$\sigma$ Standard deviation

## INTRODUCTION

The basic equations governing the incompressible creeping flow are the Stokes equations, namely,

$$
\left.\begin{array}{lllll}
\nabla^{2} \underline{v}=\nabla p & \text { or } & v_{i, j j}=p_{, i} & \text { in } & \Omega  \tag{1}\\
\nabla \cdot \underline{v}=0 & \text { or } & v_{i, i}=0 & \text { in } & \Omega
\end{array}\right\}
$$

where the quantities $\underline{v}$ and $p$ denote the dimensionless fluid velocity and the associated dimensionless pressure, respectively. In two-dimensions, the introduction of a streamfunction $\psi$ reduces the Stokes eqns (1) to the biharmonic equation

$$
\begin{equation*}
\nabla^{4} \psi=0 \quad \text { in } \Omega \tag{2}
\end{equation*}
$$

or, equivalently to the coupled system of equations

$$
\begin{equation*}
\nabla^{2} \psi=w, \quad \nabla^{2} w=0 \quad \text { in } \quad \Omega \tag{3}
\end{equation*}
$$

Inverse biharmonic boundary value problems with an underspecified boundary section being compensated for by fluid vorticity measurements at another over-specified part of the boundary have been studied in Lesnic et al. (1997). However, in practical problems such extra information has to come from measurements and frequently it is easier to measure the fluid pressure $p$ rather than the fluid vorticity $w$. Therefore, in this paper we wish to replace the extra boundary condition on the vorticity by one on the pressure and, clearly, in this case it is more appropriate to work with the Stokes eqns (1) rather than with the biharmonic eqn.(2) which, in addition is restricted to two-dimensional flows only.

Nevertheless, the initial step in obtaining a numerical solution of such an inverse and ill-posed problem is to develop a method of solution for the corresponding direct problem. Therefore, in Zeb et al. (1998) we developed the velocity-pressure boundary integral formulation for the Stokes equations. This formulation relates the fluid velocity components and the stress vector components, with the latter quantities expressed in terms of the pressure and fluid velocity following the governing constitutive equations.

In the underlying inverse Stokes problem, we investigate the numerical solution in an open bounded domain $\Omega$ enclosed by a smooth boundary $\partial \Omega$, such that

$$
\begin{equation*}
\partial \Omega=\Gamma \cup \Gamma_{0} \tag{4}
\end{equation*}
$$

where $\Gamma_{0}$ is the under-specified boundary section and $\Gamma=$ $\partial \Omega-\Gamma_{0}$. Both the normal and tangential components of the fluid velocity vector are specified on $\Gamma$, whilst only the tangential component is given on $\Gamma_{0}$. However, this underspecification of the boundary conditions is compensated by pressure measurements over $\Gamma$ or over a section of $\Gamma$.

In section 2 we formulate the problem mathematically and generate a system of linear algebraic equations by applying the BEM in conjunction with the boundary conditions. In section 3 we briefly describe Tikhonov regularization method, which is then used to solve the resulting ill-conditioned system of linear equations in section 4. Due to the ill-posed nature of the inverse Stokes problem as described above, it is important to consider the stability of the numerical solution. Therefore, in section 5 we investigate the effect of noise on the numerical solution for the unknown values of the normal fluid velocity component and the boundary pressure by adding a random error to the input data. Perturbations in the tangential component have not been investigated because, in general, the tangential component is physically available from the no-slip condition on a solid boundary and is unlikely to contain any noise.

## MATHEMATICAL FORMULATION

The Stokes eqns (1) can be transformed into an equivalent set of boundary integral equations, see Ladyzhenskaya (1963), as follows:

$$
\begin{align*}
& \eta(x) v_{k}(x)=-\int_{\partial \Omega} u_{l}^{k}(x, y) f_{l}(y) d s \\
&+\int_{\partial \Omega} K_{k l}(x, y) v_{l}(y) d s, \quad x \in \bar{\Omega} p(x)=-\int_{\partial \Omega} q^{k}(x, y) f_{k}(y) d s  \tag{5}\\
&+\int_{\partial \Omega} L_{k}(x, y) v_{k}(y) d s, \quad x \in \Omega
\end{align*}
$$

where, in two-dimensions:
(i) $\eta(x)=1$ if $x \in \Omega$, and is to the ratio of the angle between the tangents on either side of the point $x$ and $2 \pi$ if $x \in \partial \Omega$. (ii) $u_{l}^{k}(x, y)$ and $q^{k}(x, y)$ denote the fundamental solutions Stokes eqns (1), namely,

$$
\begin{array}{r}
u_{l}^{k}(x, y)=-\frac{1}{4 \pi}\left[-\delta_{k l} \ln (r)+\frac{r_{k} r_{l}}{r^{2}}\right] \\
q^{k}(x, y)=-\frac{r_{k}}{2 \pi r^{2}} \tag{7}
\end{array}
$$

Here $\delta_{k l}$ is the Kronecker delta symbol, $x=\left(x_{1}, x_{2}\right), y=$ $\left(y_{1}, y_{2}\right), r_{m}=x_{m}-y_{m},|x-y|=\sqrt{r_{m} r_{m}}$ and $m=1,2$.
(iii) The stress force components $f_{k}$ are defined by

$$
\begin{equation*}
f_{k}(y)=\sigma_{k m}(y) n_{m}(y), \sigma_{k m}(y)=-\delta_{k m} p+v_{k, m}+v_{m, k}( \tag{8}
\end{equation*}
$$

(iv) $\underline{n}=\left(n_{1}, n_{2}\right)$ are the components of the outward normal at the point $y \in \partial \Omega$.
(v) $K_{k l}(x, y)$ and $L_{k}(x, y)$ denote

$$
\begin{array}{r}
K_{k l}(x, y)=-\frac{r_{k} r_{l} r_{m}}{\pi r^{4}} n_{m} \\
L_{k}(x, y)=-2 q_{, m}^{k} n_{m}=\frac{1}{\pi}\left[\frac{n_{k}}{r^{2}}-\frac{2 r_{k} r_{m}}{r^{4}} n_{m}\right] \tag{10}
\end{array}
$$

where $q^{k}{ }_{, m}=\frac{\partial q^{k}}{\partial x_{m}}(x, y)$.
In practice the boundary integral eqns (5) and (6) can rarely be solved analytically and therefore some form of numerical approximation is necessary. Based on the BEM we subdivide the boundary $\partial \Omega$ into a series of $M$ elements $\partial \Omega_{j}, j=\overline{1, M}$, and approximate the stress vector components $f_{k}(y)$ and the fluid velocity components $v_{k}(y)$ by their
piecewise constant values taken at the centroid $\tilde{y}_{j}$ of each boundary element $\partial \Omega_{j}$ to recast eqns (5) and (6) as follows:

$$
\begin{array}{r}
\eta(x) v_{k}(x)=\sum_{j=1}^{M} v_{l}\left(\tilde{y}_{j}\right) \int_{\partial \Omega_{j}} K_{k l}(x, y) d s \\
-\sum_{j=1}^{M} f_{l}\left(\tilde{y}_{j}\right) \int_{\partial \Omega_{j}} u_{l}^{k}(x, y) d s, \quad x \in \bar{\Omega} \\
p(x)=\sum_{j=1}^{M} v_{k}\left(\tilde{y_{j}}\right) \int_{\partial \Omega_{j}} L_{k}(x, y) d s \\
-\sum_{j=1}^{M} f_{k}\left(\tilde{y_{j}}\right) \int_{\partial \Omega_{j}} q^{k}(x, y) d s, \quad x \in \Omega \tag{12}
\end{array}
$$

where $k, l=1,2$. On applying eqns (11) and (12) at the centroid node $x \equiv \tilde{y}_{i}$ of each element $\partial \Omega_{i}$ for $i=\overline{1, M}$, we obtain the following algebraic equations:

$$
\begin{align*}
& \sum_{j=1}^{M}\left[A_{i j} v_{1 j}+B_{i j} v_{2 j}+C_{i j} f_{1 j}+D_{i j} f_{2 j}\right]=0  \tag{13}\\
& \sum_{j=1}^{M}\left[E_{i j} v_{1 j}+F_{i j} v_{2 j}+G_{i j} f_{1 j}+H_{i j} f_{2 j}\right]=0 \tag{14}
\end{align*}
$$

where $v_{1 j}=v_{1}\left(\tilde{y_{j}}\right), v_{2 j}=v_{2}\left(\tilde{y_{j}}\right), f_{1 j}=f_{1}\left(\tilde{y_{j}}\right), f_{2 j}=f_{2}\left(\tilde{y_{j}}\right)$ and

$$
\begin{array}{r}
A_{i j}=\int_{\partial \Omega_{j}} K_{11}\left(\tilde{y}_{i}, y\right) d s-\eta_{j} \delta_{i j} \\
B_{i j}=\int_{\partial \Omega_{j}} K_{12}\left(\tilde{y}_{i}, y\right) d s \\
C_{i j}=-\int_{\partial \Omega_{j}} u_{1}^{1}\left(\tilde{y}_{i}, y\right) d s, \quad D_{i j}=-\int_{\partial \Omega_{j}} u_{2}^{1}\left(\tilde{y}_{i}, y\right) d s \\
E_{i j}=\int_{\partial \Omega_{j}} K_{21}\left(\tilde{y}_{i}, y\right) d s \\
F_{i j}=\int_{\partial \Omega_{j}} K_{22}\left(\tilde{y}_{i}, y\right) d s-\eta_{j} \delta_{i j} \\
G_{i j}=-\int_{\partial \Omega_{j}} u_{1}^{2}\left(\tilde{y}_{i}, y\right) d s, \quad H_{i j}=-\int_{\partial \Omega_{j}} u_{2}^{2}\left(\tilde{y}_{i}, y\right) d s \tag{15}
\end{array}
$$

For the two-dimensional domain $\Omega$, whose boundary $\partial \Omega$ is subdivided into straight line segments $\partial \Omega_{j}$, the integral coefficients given in eqn.(15) can be evaluated analytically, see Zeb et al. (1998). The eqns (13) and (14) form a system of $2 M$ equations in $4 M$ unknowns. Clearly, if either the
fluid velocity vector or the stress vector is known on $\partial \Omega$ then the other quantity can be readily found from these equations, which in turn allows the fluid velocity and the pressure inside the domain $\Omega$ to be calculated from eqns (11) and (12). However, since the inverse Stokes problem which is considered in this paper fails to provide both fluid velocity components on the whole of the boundary, it is necessary to introduce extra pressure measurements from an over-specified section of the boundary. This additional information can then be combined with eqns (13) and (14) to provide a system of equations with which to solve the problem. For the boundaries normal to the Cartesian coordinate directions $x_{1}$ and $x_{2}$ this can be easily achieved from,

$$
\begin{align*}
& f_{1}=-p+2 e_{11}=-p-2 e_{22} \\
& f_{2}=-p+2 e_{22}=-p-2 e_{11} \tag{16}
\end{align*}
$$

where $e_{11}=\frac{\partial v_{1}}{\partial x_{1}}$ and $e_{22}=\frac{\partial v_{2}}{\partial x_{2}}$. If in addition the fluid velocity on such boundaries is given on the boundary region $x_{1}=$ constant then $e_{22}$ can be determined and hence eqns (16) relate $f_{1}$ and $f_{2}$ to the pressure, whereas $e_{11}$ is available on $x_{2}=$ constant and again eqns (16) relate $f_{1}$ and $f_{2}$ to the pressure. However, in geometrical situations where the boundaries are not parallel to either the $x_{1}$ or the $x_{2}$ axes, the boundaries can be expressed in a new localised coordinate system $\left(\bar{x}_{1}, \bar{x}_{2}\right)$ by $\bar{x}_{1}=$ constant. Then it is more natural to consider components of the fluid velocity vector ( $\bar{v}_{1}, \bar{v}_{2}$ ) normal and tangential to the boundary. In terms of the coordinates $\left(\bar{x}_{1}, \bar{x}_{2}\right)$, as opposed to ( $x_{1}, x_{2}$ ), the inverse Stokes problem can then be mathematically stated as follows:

$$
\left\{\begin{array}{lll}
\nabla^{2} \underline{v} & =\nabla p &  \tag{17}\\
\text { in } \Omega \\
\nabla \cdot \underline{v} & =0 & \\
\text { in } \Omega \\
\bar{v}_{2}\left(\bar{x}_{1}, \bar{x}_{2}\right) & =\phi\left(\bar{x}_{2}\right) & \\
\text { on } \partial \Omega \\
\bar{v}_{1}\left(\bar{x}_{1}, \bar{x}_{2}\right) & =\chi\left(\bar{x}_{2}\right) & \\
\text { on } \Gamma \\
p\left(\bar{x}_{1}, \bar{x}_{2}\right) & =\xi\left(\bar{x}_{2}\right) & \\
\text { on } \Gamma^{\star}
\end{array}\right.
$$

where $\Gamma^{\star} \subseteq \Gamma$ and $\phi, \chi$ and $\xi$ are given functions of $\bar{x}_{2}$. Therefore, using the coordinate transformation from $\left(x_{1}, x_{2}\right)$ to ( $\bar{x}_{1}, \bar{x}_{2}$ ), we express the fluid velocity components $\left(v_{1}, v_{2}\right)$ in terms of $\left(\bar{v}_{1}, \bar{v}_{2}\right)$ through the relation,

$$
\begin{equation*}
v_{j}=l_{i j} \bar{v}_{i} \tag{18}
\end{equation*}
$$

where $l_{i j}$ are the direction cosines of $\bar{x}_{i}$ axis with respect to $x_{j}$ axis and depend on the position of the element $\partial \Omega_{j}$, i.e. $l_{i j}=l_{i j}\left(\tilde{y}_{j}\right)=l_{i j}\left(\tilde{\bar{y}}_{j}\right)$, where $\tilde{y}_{j}$ and $\tilde{\bar{y}}_{j}$ denote the mid point
of $\partial \Omega_{j}$ in $\left(x_{1}, x_{2}\right)$ and ( $\left.\bar{x}_{1}, \bar{x}_{2}\right)$ coordinate systems, respectively. However, the notation for the dependence of $l_{i j}$ on $\tilde{y}_{j}$ or $\tilde{\bar{y}}_{j}$ is omitted for the sake of convenience. Introducing eqn.(18) into eqns (13) and (14) for $i=\overline{1, M}$, results in,

$$
\begin{align*}
& \sum_{j=1}^{M}\left[A_{i j}^{*} \bar{v}_{1 j}+B_{i j}^{*} \bar{v}_{2 j}+C_{i j} f_{1 j}+D_{i j} f_{2 j}\right]=0  \tag{19}\\
& \sum_{j=1}^{M}\left[E_{i j}^{*} \bar{v}_{1 j}+F_{i j}^{*} \bar{v}_{2 j}+G_{i j} f_{1 j}+H_{i j} f_{2 j}\right]=0 \tag{20}
\end{align*}
$$

where

$$
\left.\begin{array}{l}
A_{i j}^{*}=A_{i j} l_{11}+B_{i j} l_{12}, B_{i j}^{*}=A_{i j} l_{21}+B_{i j} l_{22}  \tag{21}\\
E_{i j}^{*}=E_{i j} l_{11}+F_{i j} l_{12}, F_{i j}^{*}=E_{i j} l_{21}+F_{i j} l_{22}
\end{array}\right\}
$$

Moreover, making use of the constitutive equations and the equation of continuity, with the fluid velocity and the stress tensor referred to the new co-ordinate system ( $\bar{x}_{1}, \bar{x}_{2}$ ), we obtain

$$
\begin{equation*}
\bar{f}_{1}=-\bar{p}+2 \bar{e}_{11}=-\bar{p}-2 \bar{e}_{22} \tag{22}
\end{equation*}
$$

Now using the definition of the rate of strain tensor in terms of the velocity vector $\underline{\bar{v}}=\left(\bar{v}_{1}, \bar{v}_{2}\right)$, eqn.(22) can be re-expressed as follows:

$$
\begin{equation*}
\bar{f}_{1}=-\bar{p}-2\left[\frac{1}{\bar{h}_{2}} \frac{\partial \overline{\underline{v}}_{2}}{\partial \bar{x}_{2}}+\frac{\overline{\underline{v}}_{1}}{\bar{h}_{1} \bar{h}_{2}} \frac{\partial \bar{h}_{2}}{\partial \bar{x}_{1}}\right] \tag{23}
\end{equation*}
$$

where $\bar{h}_{1}, \bar{h}_{2}$ are the scale factors for the coordinate system $\left(\bar{x}_{1}, \bar{x}_{2}\right)$. Whilst it is intended to employ the fluid velocity components in terms of the new coordinate system $\left(\bar{x}_{1}, \bar{x}_{2}\right)$, the stress tensor is retained in the original Cartesian coordinate system ( $x_{1}, x_{2}$ ), and hence eqn.(23) can be re-expressed as follows:

$$
\begin{equation*}
\bar{f}_{1} \equiv l_{1 j} f_{j}=-\bar{p}-2\left[\frac{1}{\bar{h}_{2}} \frac{\partial \overline{\underline{v}}_{2}}{\partial \bar{x}_{2}}+\frac{\overline{\underline{v}}_{1}}{\bar{h}_{1} \bar{h}_{2}} \frac{\partial \bar{h}_{2}}{\partial \bar{x}_{1}}\right] \tag{24}
\end{equation*}
$$

Now suppose that $M=M_{0}+M_{1}+M_{2}$, where $M_{0}$ belong to $\Gamma_{0}, M_{1}$ to $\Gamma^{\star}$ and the remaining $M_{2}$ belong to $\Gamma-\Gamma^{\star}$, then discretising eqn.(24) we see that the pressure boundary condition over $\Gamma^{\star}$ for $i=\overline{1, M}$, gives

$$
\begin{equation*}
\sum_{j=1}^{M}\left[T_{i j} \underline{f}_{1 j}+W_{i j} \underline{f}_{2 j}\right]=-\bar{p}_{i}-2\left[\frac{v_{2 i, 2}}{\bar{h}_{2 i}}+\frac{\bar{v}_{1 i} \bar{h}_{2 i, 1}}{\bar{h}_{1 i} \bar{h}_{2 i}}\right] \tag{25}
\end{equation*}
$$

where $\bar{h}_{1 i}=\bar{h}_{1}\left(\tilde{\bar{y}}_{i}\right), \bar{h}_{2 i}=\bar{h}_{2}\left(\tilde{\bar{y}}_{i}\right), \bar{v}_{1 i}=\bar{v}_{1}\left(\tilde{\bar{y}}_{i}\right), \bar{h}_{2 i, 1}=$ $\frac{\partial \bar{h}_{2}}{\partial \bar{x}_{1}}\left(\tilde{\bar{y}}_{i}\right), \bar{v}_{2 i, 2}=\frac{\partial \bar{v}_{2 i}}{\partial \bar{x}_{2}}\left(\tilde{\bar{y}}_{i}\right)$ and

$$
\left.\begin{array}{ll}
T_{i j}=0, \quad W_{i j}=0 & \text { if } i \neq j  \tag{26}\\
T_{i j}=l_{11}, W_{i j}=l_{12} & \text { if } i=j
\end{array}\right\}
$$

Then eqns (19), (20) and (25) provide a complete set of $2 M+M_{1}$ equations involving $4 M$ unknowns, namely the fluid velocity components $\left(\bar{v}_{1}, \bar{v}_{2}\right)$ and the stress vector components $\left(f_{1}, f_{2}\right)$, if $p$ is considered to be known. Applying the boundary conditions on $\bar{v}_{1}$ and $\bar{v}_{2}$, we can eliminate $M+M_{1}+M_{2}$ unknowns and the inverse problem (17) reduces to solving the following system of $2 M+M_{1}$ equations in $2 M+M_{0}$ unknowns,

$$
\begin{equation*}
\mathbb{A} \underline{x}=\underline{b} \tag{27}
\end{equation*}
$$

where $\mathbb{A}$ is a $\left(2 M+M_{1}\right) \times\left(2 M+M_{0}\right)$ matrix depending on the integral matrices $A^{*}, C, D, E^{*}, G$ and $H, \underline{x}=$ $\left[\left.\underline{f}_{1}\right|_{\partial \Omega},\left.\underline{f}_{2}\right|_{\partial \Omega},\left.\underline{\bar{v}}_{1}\right|_{\Gamma_{0}}\right]^{t}$ is the required solution vector having $2 \bar{M}+\bar{M}_{0}$ components and $\underline{b}$ is the known vector of order $2 M+M_{1}$.

## REGULARIZATION METHOD

The Tikhonov regularization method is one of the most powerful and efficient methods for solving inverse and illposed problems which arise in science and engineering. It modifies the least squares approach and finds an approximate numerical solution which in the case of the zero-th order regularization procedure is given by, see Tikhonov and Arsenin (1977),

$$
\begin{equation*}
\underline{x}_{\lambda}=\left(\mathbb{A}_{\mathbb{A}^{t}}^{t}+\lambda I\right)^{-1} \mathbb{A}^{t} \underline{b} \tag{28}
\end{equation*}
$$

where $I$ is the identity matrix and $\lambda>0$ is the regularization parameter, which controls the degree of smoothing applied to the solution and whose choice may be based on the L-curve method, see Hansen (1992). For the zero-th order regularization procedure we plot on a log-log scale the variation of $\left\|\underline{x}_{\lambda}\right\|$ versus the residual $\left\|\mathbb{A} \underline{x}_{\lambda}-\underline{b}\right\|$ for a wide range of values of $\lambda>0$. In many applications this graph results in a L-shaped curve and the choice of the optimal regularization parameter $\lambda>0$ is based on selecting approximately the corner of this $L$-curve.

## NUMERICAL RESULTS AND DISCUSSION

In section 3 we described the numerical scheme used to obtain a stable numerical solution of the inverse Stokes
problem (17) and in this section we illustrate the numerical results obtained by that scheme. In order to avoid any corner singularities we consider a smooth geometry in the form of a circle of radius $R=2$, in which case ( $\bar{x}_{1}, \bar{x}_{2}$ ) are identified as the polar coordinates $(r, \theta)$. Further the fluid velocity components ( $\bar{v}_{1}, \bar{v}_{2}$ ) in eqns (19) and (20) are replaced by $\left(v_{r}, v_{\theta}\right)$ and the scale factors ( $\bar{h}_{1}, \bar{h}_{2}$ ) in eqns (24) and (25) by $(1, r)$. We consider a situation where no analytical solution is available and in that case the numerical solution of the inverse Stokes problem (17) is compared with the solution obtained by solving numerically the corresponding direct problem using the BEM.

In the inverse problem (17) we choose $\Gamma^{\star}=\{(R, \theta)$ : $\left.\theta_{1}<\theta<\theta_{2}\right\}$ and $\Gamma_{0}=\left\{(R, \theta): \theta_{3}<\theta<\theta_{4}\right\}$. We fix $\left(\theta_{1}, \theta_{2}\right)$ at $\left(0, \frac{\pi}{4}\right)$ and move $\left(\theta_{3}, \theta_{4}\right)$ to different positions in order to study the effects of various locations for the under-specified boundary region. First we keep the length of $\Gamma_{0}$ equal to that of $\Gamma^{\star}$ and then reduce its length by one half. Moreover, taking advantage of the symmetry about the line $\theta=\pi / 4$, we take $\left(\theta_{3}, \theta_{4}\right)$ at $\left(\frac{\pi}{4}, \frac{\pi}{2}\right),\left(\frac{\pi}{2}, \frac{3 \pi}{4}\right)$, $\left(\frac{3 \pi}{4}, \pi\right)$ and $\left(\pi, \frac{5 \pi}{4}\right)$ and, to halve the length of $\Gamma_{0}$ we replace each of these intervals by $\left(\frac{5 \pi}{16}, \frac{7 \pi}{16}\right),\left(\frac{9 \pi}{16}, \frac{11 \pi}{16}\right),\left(\frac{13 \pi}{16}, \frac{15 \pi}{16}\right)$ and $\left(\frac{17 \pi}{16}, \frac{19 \pi}{16}\right)$, respectively.

The functions $\phi$, as given in eqn.(17), is chosen such that $\phi=0$ which physically represents the no-slip condition, whilst the function $\chi$ is taken to be

$$
\chi=\left\{\begin{array}{ccc}
\Theta_{1}\left(\theta-\theta_{1}\right)\left(\theta-\theta_{2}\right) & \text { over } & \Gamma^{\star}  \tag{29}\\
0 & \text { over } & \Gamma-\Gamma^{\star}
\end{array}\right.
$$

It is worth noting here that if the constant $\Theta_{1}$ takes a positive value then $\left.\chi\right|_{\Gamma^{\star}}<0$, which means that the fluid flows into the domain $\Omega$ through the boundary region $\Gamma^{\star}$. In practical situations the value of the function $\xi$ on $\Gamma^{\star}$ has to come from pressure measurements. However, in the present study we first consider a direct problem formulation in which $v_{r}$ is specified over $\Gamma_{0}$ as,

$$
\begin{equation*}
v_{r}=\Theta_{2}\left(\theta-\theta_{3}\right)\left(\theta-\theta_{4}\right) \quad \text { over } \quad \Gamma_{0} \tag{30}
\end{equation*}
$$

Since from continuity the fluid has to flow out of the domain $\Omega$ through the boundary region $\Gamma_{0}$ the constant, $\Theta_{2}$ must take a negative value to ensure that $\left.v_{r}\right|_{\Gamma_{0}}>0$. Moreover, as required by mass conservation, the inflow rate through $\Gamma^{\star}$ is equal to the outflow rate through $\Gamma_{0}$, i.e.

$$
\int_{\Gamma^{\star}} \Theta_{1}\left(\theta-\theta_{1}\right)\left(\theta-\theta_{2}\right) d \theta=\int_{\Gamma_{0}} \Theta_{2}\left(\theta-\theta_{3}\right)\left(\theta-\theta_{4}\right) d \theta(31)
$$

Thus, in view of the above two requirements, the constants
$\Theta_{1}$ and $\Theta_{2}$ are related through the relation

$$
\begin{equation*}
\Theta_{2}=-\Theta_{1}\left(\frac{\theta_{2}-\theta_{1}}{\theta_{4}-\theta_{3}}\right) \tag{32}
\end{equation*}
$$

Solving this direct problem with $v_{r}$ and $v_{\theta}$ known on $\partial \Omega$, we obtain the pressure $p$ everywhere and, in particular, over $\Gamma^{\star}$. This numerically calculated pressure, denoted by $p_{n}$, is used as the value of the function $\xi$ in the inverse problem (17) in which $v_{r}$ is assumed unknown on $\Gamma_{0}$. If we fix $\Theta_{1}=1$ then it is clear from eqn. (32) that

$$
\Theta_{2}=\left\{\begin{array}{lll}
-1 & \text { if } & \text { length of } \Gamma_{0}=\text { length of } \Gamma^{\star}  \tag{33}\\
-2 & \text { if } & \text { length of } \Gamma_{0}=\frac{1}{2}\left(\text { length of } \Gamma^{\star}\right)
\end{array}\right.
$$

Whilst in the direct Stokes problem we observed that the difference between the numerical results for $p$ using $M=160$ and 320 was less than $1 \%$, in the inverse problem (17) which follows, we found that $M=160$ was sufficiently large for the numerical solution to agree graphically with the corresponding numerical solution from the direct problem.

Figure 1 shows the numerical solution for the unspecified values of the normal component of the fluid velocity $v_{r}$ over $\Gamma_{0}=\left\{(R, \theta): \frac{\pi}{4}<\theta<\frac{\pi}{2}\right\}$ for $\lambda=10^{-15}$, together with its values specified in the corresponding direct problem. From this figure it is observed that the agreement between the numerical solution and the one given in eqn.(30), which is specified analytically over $\Gamma_{0}$ in the direct problem, is excellent.

In Fig.2, we present the numerical solution for the associated boundary pressure $p$ over $\partial \Omega-\Gamma^{\star}$ obtained using the numerical solutions for $f_{1}, f_{2}$ and $\left.v_{r}\right|_{\Gamma_{0}}$, together with the known boundary data for $\left.v_{r}\right|_{\Gamma}$ and $\left.v_{\theta}\right|_{\partial \Omega}$. Also included in this figure is the boundary pressure obtained from the solution of the direct problem with $v_{\theta}=0$ on $\partial \Omega$ and with $v_{r}$ over $\Gamma$ and $\Gamma_{0}$ as given in eqns (29) and (30), respectively. It can be seen from Fig. 2 that the numerical solution generated by the inverse problem agrees well with the corresponding numerical solution obtained from the direct problem except at the points where the curvature of the solution is large.

In order to visualise the overall flow pattern inside the circular domain $\Omega$ we present in Fig. 3 the velocity vectors at selected points for each of the four locations for the underspecified boundary portion $\Gamma_{0}$. The lengths of the vectors and of the arrows are proportional to the magnitude of the fluid velocity. Although not illustrated graphically, we wish to report that both the magnitude and the direction of the fluid velocity vectors were observed to be similar to those


Figure 1. THE NUMERICAL SOLUTION FOR THE NORMAL COMPONENT OF THE FLUID VELOCITY $\left.v_{r}\right|_{\Gamma_{0}}$, TOGETHER WITH THE VALUES OF $v_{r}$ ANALITYCALLY SPECIFIED OVER $\partial \Omega$ IN THE CORRESPONDING DIRECT PROBLEM, WHEN $\Gamma_{0}=\left\{(R, \theta): \frac{\pi}{4}<\theta<\frac{\pi}{2}\right\}$ AND $\lambda=10^{-15}$.


Figure 2. THE NUMERICAL SOLUTION FOR THE BOUNDARY PRESSURE $\left.p\right|_{\partial \Omega-\Gamma^{*}}$, TOGETHER WITH THE CORRESPONDING NUMERICAL SOLUTION FOR $p$ OVER $\partial \Omega$ IN THE DIRECT PROBLEM, WHEN $\Gamma_{0}=\left\{(R, \theta): \frac{\pi}{4}<\theta<\frac{\pi}{2}\right\}$ AND $\lambda=10^{-15}$.
obtained in the direct problem. Also it is reported that the numerical results for the lines of constant pressure for each of the four locations for $\Gamma_{0}$, which are obtained by
solving the direct and the inverse problems were found to be undistinguishable.

Next we move to the situation where the length of $\Gamma_{0}$ is one half that of the over-specified boundary portion $\Gamma^{\star}$. Without presenting the results graphically, it is reported that agreement of the numerical results for the normal fluid velocity $\left.v_{r}\right|_{\Gamma_{0}}$ and the boundary pressure $\left.p\right|_{\partial \Omega-\Gamma^{\star}}$ for all the four possible locations of $\Gamma_{0}$, with halved length and with $\lambda=10^{-15}$, was found to be equivalent to that observed in Figs 1 and 2, respectively. Figure 4 shows the fluid velocity vectors at selected interior points when the length of the under-specified region $\Gamma_{0}$ is halved. Although not illustrated, we wish to report that both the magnitude and the direction of these fluid velocity vectors are found to be similar to those obtained from the direct problem solution.


Figure 3. FLUID VELOCITY VECTORS AT SELECTED POINTS INSIDE THE CIRCULAR DOMAIN $\Omega$ OBTAINED BY SOLVING THE INVERSE PROBLEM (17) FOR VARIOUS LOCATIONS OF $\Gamma_{0}$, NAMELY: (a) $\Gamma_{0}=$ $\left\{(R, \theta): \frac{\pi}{4}<\theta<\frac{\pi}{2}\right\}$, (b) $\Gamma_{0}=\left\{(R, \theta): \frac{\pi}{2}<\theta<\frac{3 \pi}{4}\right\}$, (c) $\Gamma_{0}=\left\{(R, \theta): \frac{3 \pi}{4}<\theta<\pi\right\}$ AND (d) $\Gamma_{0}=\left\{(R, \theta): \pi<\theta<\frac{5 \pi}{4}\right\}$.


Figure 4. FLUID VELOCITY VECTORS AT SELECTED POINTS INSIDE THE CIRCULAR DOMAIN $\Omega$ OBTAINED BY SOLVING THE INVERSE PROBLEM (17) FOR VARIOUS LOCATIONS OF $\Gamma_{0}$, NAMELY: (a) $\Gamma_{0}=$ $\left\{(R, \theta): \frac{5 \pi}{16}<\theta<\frac{7 \pi}{16}\right\}$, (b) $\Gamma_{0}=\left\{(R, \theta): \frac{9 \pi}{16}<\theta<\frac{11 \pi}{16}\right\}$, (c) $\Gamma_{0}=\left\{(R, \theta): \frac{13 \pi}{16}<\theta<\frac{15 \pi}{16}\right\}$ AND (d) $\Gamma_{0}=\left\{(R, \theta): \frac{17 \pi}{4}<\theta<\right.$ $\left.\frac{19 \pi}{2}\right\}$.

## EFFECT OF NOISE

As mentioned in the introduction, since the inverse Stokes problem (17) is ill-posed the system of eqns (27) that results by the application of a boundary element discretization is ill-conditioned and the solution may not continuously depend upon the input data. Therefore the stability of the regularized boundary element technique is investigated in this section by adding small amounts of random noise into the input data in order to simulate measurement errors which are innately present in a data set of any practical problem. Hence, we perturb the given boundary data by adding a random noisy variables $\epsilon$ to the functions $\xi$ and $\chi$ as given in eqn.(17), namely,

$$
\begin{equation*}
\bar{\xi}=\xi+\epsilon, \quad \text { and } \quad \bar{\chi}=\chi+\epsilon \tag{34}
\end{equation*}
$$

The random errors $\epsilon$ represent Gaussian random variables of mean zero and standard deviation $\sigma$, which is taken to be some percentage $\alpha$ of the maximum value of $p$ or $v_{r}$ i.e.

$$
\begin{equation*}
\sigma=\max |p| \times \frac{\alpha}{100} \quad \text { or } \quad \sigma=\max \left|v_{r}\right| \times \frac{\alpha}{100} \tag{35}
\end{equation*}
$$

For a particular location of $\Gamma_{0}$, say $\Gamma_{0}=\left\{(R, \theta): \frac{\pi}{2}<\right.$ $\left.\theta<\frac{3 \pi}{4}\right\}$, we observed that the corner of the $L$-curve graphs of the solution norm $\left\|\underline{x}_{\lambda}\right\|$ versus the norm of the residual vector $\left\|\stackrel{A}{\underline{x}_{\lambda}}-\underline{b}\right\|$ for various amounts of noise $\alpha \in\{1,3,5\}$ introduced in $\left.p\right|_{\Gamma^{\star}}$ correspond to the following values of $\lambda$,

$$
\lambda_{\text {opt }} \approx\left\{\begin{array}{l}
1 \times 10^{-7} \text { if } \alpha=1  \tag{36}\\
1 \times 10^{-6} \text { if } \alpha=3 \\
5 \times 10^{-6} \text { if } \alpha=5
\end{array}\right.
$$

and, hence, these values are postulated to be the appropriate choice for $\lambda$. Further, when we replace $\left(\theta_{3}, \theta_{4}\right)=\left(\frac{\pi}{2}, \frac{3 \pi}{4}\right)$ by $\left(\theta_{3}, \theta_{4}\right)=\left(\frac{9 \pi}{16}, \frac{11 \pi}{16}\right)$ to halve the length of $\Gamma_{0}$, the corresponding $L$-curve graphs gave the same optimal values of $\lambda$ as given in eqn.(36).

It was found that the numerical solution for the normal fluid velocity $v_{r}$ over $\Gamma_{0}=\left\{(R, \theta): \frac{\pi}{2}<\theta<\frac{3 \pi}{4}\right\}$ and $\left\{(R, \theta): \frac{9 \pi}{16}<\theta<\frac{11 \pi}{16}\right\}$ obtained using $\lambda_{\text {opt }}$ given in eqn.(36) remains stable and agrees with the values of $\left.v_{r}\right|_{\Gamma_{0}}$ specified in the direct problem reasonably well according to the amount of noise introduced in the input data for $\left.p\right|_{\Gamma^{\star}}$. Moreover, the errors in the numerical solution for $\left.p\right|_{\partial \Omega-\Gamma^{*}}$ in comparison with the the corresponding numerical solution generated by solving the direct problem were found to be comparable with the amount of noise in the boundary data $\left.p\right|_{\Gamma^{\star}}$. Therefore, omitting the boundary results, we present in Figs 5 and 6 the lines of constant pressure inside the domain $\Omega$ for $\Gamma_{0}=\left\{(R, \theta): \frac{\pi}{2}<\theta<\frac{3 \pi}{4}\right\}$ and $\left\{(R, \theta): \frac{9 \pi}{16}<\theta<\frac{11 \pi}{16}\right\}$, respectively, obtained using the given and computed boundary data when $0 \%, 1 \%, 3 \%$ and $5 \%$ noise was included in $\left.p\right|_{\Gamma^{\star}}$. Also included in these figures is the interior numerical solution obtained from the corresponding direct problem. It is observed from Figs 5 and 6 that as $\alpha$ decreases from 5 to 0 then the interior numerical solution for the pressure $p$ generated by solving the inverse problem approaches, whilst remaining stable, the numerical solution obtained from the direct problem. Also a similar behaviour of the numerical solution was observed when the input boundary data for $\left.v_{r}\right|_{\Gamma^{\star}}$ was perturbed. This demonstrates the stability of the solution both on the boundary and inside the solution domain.

Overall from the above discussion it is concluded that the numerical solution of the inverse problem (17) produced by the BEM combined with the regularization method for


Figure 5. THE LINES OF CONSTANT PRESSURE $p$ INSIDE THE CIRCULAR DOMAIN $\Omega$ WHEN $\Gamma_{0}=\left\{(R, \theta): \frac{\pi}{2}<\theta<\frac{3 \pi}{4}\right\}$ AND VARIOUS LEVELS OF NOISE INTRODUCED IN $\left.p\right|_{\Gamma^{\star}}$, NAMELY, DIRECT SOLUTION (一) AND INVERSE SOLUTION WITH $\alpha=0(\cdots), \alpha=1(---)$, $\alpha=3(-\cdot-)$ AND $\alpha=5(-\cdots-)$.
the value of $\lambda_{\text {opt }}$ chosen at the corner of the $L$-curve, is accurate, stable and convergent with respect to decreasing the amount of noise.

## CONCLUSIONS

In this paper the Stokes equations which govern the slow viscous flow of fluids, subject to under-specified boundary conditions on the normal component of the fluid velocity but with additional pressure measurements available on another portion of the boundary, have been studied. A boundary element discretization has been applied to the Stokes equations and the resulting ill-conditioned system of linear equations solved using the Tikhonov regularization method. The technique has been validated for a typical benchmark test examples in a circular domain in a situation where no analytical solution is available. It has been concluded that this regularized boundary element technique retrieves an accurate, stable and convergent numerical solution, both on the boundary and inside the domain, with respect to increasing the number of boundary elements and decreasing the amount of noise in the input data.


Figure 6. THE LINES OF CONSTANT PRESSURE $p$ INSIDE THE CIRCULAR DOMAIN $\Omega$ WHEN $\Gamma_{0}=\left\{(R, \theta): \frac{9 \pi}{16}<\theta<\frac{11 \pi}{16}\right\}$ AND VARIOUS LEVELS OF NOISE INTRODUCED IN $\left.p\right|_{\Gamma^{\star}}$, NAMELY, DIRECT SOLUTION (-) AND INVERSE SOLUTION WITH $\alpha=0(\cdots), \alpha=1$ $(---), \alpha=3(-\cdots-)$ AND $\alpha=5(-\cdots-)$.

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