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ITERATIVE METHODS FOR SOLVING INVERSE COEFFICIENT PROBLEM FOR STATIONARY HEAT CONDUCTION EQUATION

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ABSTRACT

Boundary value problem for one dimensional stationary heat conduction equation with unknown heat conductivity coefficient was considered. One of the boundary conditions depended on parameter. Heat flux was used for determining the unknown coefficient. Two iterative methods to solve this inverse problem were suggested.

INTRODUCTION

Inverse coefficient problems for nonlinear heat conduction equation arise in various directions of science and engineering (see as example [1]) . It is very important to underline that these inverse problems have strong nonlinearity. There are some approaches for construction of iterative methods to solve them. One consists in application of iteration scheme for realization of Tikhonov regularization method [2]. Another uses a special finite-dimensional representation for unknown coefficient. In this method we reduce inverse problem to a finite-dimensional minimization problem and apply an iterative method for its solution [1].

In this work we give examples of iterative methods closely connected with specific features of inverse problem. We consider inverse problem for one-dimensional stationary heat conduction equation with unknown heat conduction coefficient. Boundary condition depends on parameter. Inverse coefficient problems for nonlinear ordinary differential equations with parameter have been studied in [3],[4],[5].

NOMENCLATURE

k = heat conduction coefficient
f = heat source
u = temperature
s = parameter

We consider the following boundary value problem for one dimensional stationary heat conduction equation

$$(k(u)u')' = f(x,u),\ 0 < x < 1, \tag{1}$$

$$u(0) = 0, \tag{2}$$

$$u(1) = s, \tag{3}$$

where function $f(x,y)$ is given, positive function $k(y)$ is unknown and s is nonnegative parameter. We explore the following inverse problem: to determine heat conduction coefficient $k(y)$, if additional information on solution of boundary value problem (1)-(3)

$$k(u(0))u'(0) = b(s), \tag{4}$$

is given for all values of parameter s from segment $[0,s_0]$. The main goal of this paper consists in construction of iterative methods for solving the inverse problem.

We assume that function $f(x,y)$ has continuous first partial derivatives for $x \in [0,1], y \in [0,s_0]$, $f(x,0) = 0$, $f(x,y) > 0$ for $x \in [0,1], y \in (0,s_0]$, function $b(s)$ has continuous positive derivative for $s \in [0,s_0]$ and $b(0) = 0$.

Further we consider a solution of the problem (1)-(3) $u(x,s)$ as function of two variables x and s. Let

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functions $k(y)$ and $u(x,y)$ solve equations (1)-(4). Integrating equation (1) and using (4), we have

$$k(u(x,s))u'(x,s) = b(s) + \int_0^x f(z,u(z,s))dz \qquad (5)$$

From (2), (5) we obtain

$$\int_0^{u(x,s)} k(y)dy = xb(s) + \int_0^x f(z,u(z,s))(x-z)dz \qquad (6)$$

Let $p(x,s)$ be the first derivative of $u(x,s)$ with respect to parameter s. Differentiating (6) with respect to s, we get the integral equation for function $p(x,s)$

$$k(u(x,s))p(x,s) = xb'(s) + \int_o^x f_y(z,u(z,s))p(z,s)(x-z)dz \qquad (7)$$

Setting $x=1$ and using (3), we have

$$k(s) = b'(s) + \int_0^1 (1-z) f_y(z,u(z,s))p(z,s)dz \qquad (8)$$

Using equations (5),(7),(8), we present iterative method for calculation of unknown functions $k(y), u(x,s), p(x,s)$. Let function $k_n(y)$ be given. Then $u_n(x,s)$ is solution of nonlinear integral equation

$$u_n(x,s) = b(s)\int_0^x (k_n(u_n(z,s)))^{-1}dz + \int_0^x (k_n(u_n(z,s)))^{-1}\int_0^z f(t,u_n(t,s))dtdz \qquad (9)$$

For given functions $k_n(y)$ and $u_n(x,s)$ function $p_n(x,s)$ solves linear integral equation

$$k_n(u_n(x,s))p_n(x,s) = xb'(s) + \int_o^x f_y(z,u_n(z,s))p_n(z,s)(x-z)dz \qquad (10)$$

Finally function $k_{n+1}(y)$ is defined by formula

$$k_{n+1}(s) = b'(s) + \int_0^1 (1-z) f_y(z,u_n(z,s))p_n(z,s)dz \qquad (11)$$

Therefore (9)-(11) define the iterative process for solving inverse problem.

Now we consider more simple situation, when function f does not depend on x: $f(x,y) = f(y)$. In this case it is possible to obtain a nonlinear integral equation for unknown function $k(y)$ only. Multiplying (1) by $2k(u)u'$ and integrating from 0 to x, we have

$$(k(u(x,s))u'(x,s))^2 = b(s)^2 + 2\int_0^x f(u(z,s))k(u(z,s))u'(z,s)dz$$

Making change of variable in integral and using condition (2), we obtain

$$k(u(x,s))u'(x,s) \times \left[1 + 2b(s)^{-2}\int_0^{u(x,s)} f(y)k(y)dy\right]^{-\frac{1}{2}} = b(s)$$

Integration of this equation from 0 to 1 yields

$$\int_0^s k(t)\left[1 + 2b(s)^{-2}\int_0^t f(y)k(y)dy\right]^{-\frac{1}{2}} dt = b(s)$$

Differentiating with respect to s, we have

$$k(s)\left[1 + 2b(s)^{-2}\int_0^s f(y)k(y)dy\right]^{-\frac{1}{2}} + 2\int_0^s k(t)\left[1 + 2b(s)^{-2}\int_0^t f(y)k(y)dy\right]^{-\frac{3}{2}} \times b(s)^{-3}b'(s)\int_0^t f(y)k(y)dydt = b'(s)$$

or

$$k(s) = \left[1 + 2b(s)^{-2}\int_0^s f(y)k(y)dy\right]^{\frac{1}{2}} \times \left\{b'(s) - 2\int_0^s k(t)\left[1 + 2b(s)^{-2}\int_0^t f(y)k(y)dy\right]^{-\frac{3}{2}} \times b(s)^{-3}b'(s)\int_0^t f(y)k(y)dydt\right\} \qquad (12)$$

This equation is the nonlinear integral equation for unknown function $k(y)$. It is not difficult to write the iterative process for solving equation (12)

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$$k_{n+1}(s)=\left[1+2b(s)^{-2}\int_0^s f(y)k_n(y)dy\right]^{\frac{1}{2}}$$
$$\times\left\{b'(s)-2\int_0^s k_n(t)\left[1+2b(s)^{-2}\int_0^t f(y)k_n(y)dy\right]^{-\frac{3}{2}}\right.$$
$$\left.\times b(s)^{-3}b'(s)\int_0^t f(y)k_n(y)dydt\right\} \quad (13)$$

There is a big difference between iterative process (9)-(11) and iterative process (13). In the first one we calculate successive approximations $k_{n+1}(s)$ for unknown function $k(s)$ from $k_n(s), u_n(x,s)$ and $p_n(x,s)$. In the iterative process (13) we determine $k_{n+1}(s)$ using only function $k_n(s)$.

Remark. Equations (10),(11) and (13) involve first derivative $b'(s)$ of function $b(s)$. If function $b(s)$ is given only approximately, then we have to apply regularization methods [2] for stable calculation of $b'(s)$.

NUMERICAL EXAMPLE

Now we consider a numerical example of application of the iterative method (13) for solving the inverse problem.

The boundary value problem (1)-(3) has been solved for

$$k(y)=1+2y^2, \quad f(y)=\begin{cases} y^2, & 0\le y\le 1, \\ 2y-1, & y\ge 1. \end{cases}$$

$s\in[0,2]$ and functions $b(s)$, $b'(s)$ were calculated. Then iterative method (13) was used to solve inverse problem.. Function $k_1(s)=1$ was used as first approximation of iterative method (13). Approximate solution $k_a(s)$ was obtained. Functions $k_a(s)$ and $k(s)=1+2s^2$ are given in Table

s	k_a(s)	k(s)	s	k_a(s)	k (s)
0.00	1.000	1.000	1.00	2.986	3.000
0.10	1.018	1.020	1.10	3.404	3.420
0.20	1.077	1.080	1.20	3.863	3.880
0.30	1.176	1.180	1.30	4.362	4.380
0.40	1.314	1.320	1.40	4.899	4.920
0.50	1.493	1.500	1.50	5.477	5.500
0.60	1.712	1.720	1.60	6.095	6.120
0.70	1.970	1.980	1.70	6.752	6.780
0.80	2.269	2.280	1.80	7.451	7.480
0.90	2.607	2.620	1.90	8.188	8.220
1.00	2.986	3.000	2.00	8.965	9.000

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REFERENCES

1.Alifanov, O.M., Artyukhin, E.A., and Rumyantsev, S.V., 1995, «Extreme Methods for Solving Ill-Posed Problems with Applications to Inverse Problems», Bejell House.

2.Tikhonov, A.N., and Arsenin, V.Ya., 1977, «Solution of Ill-Posed Problems», John Wiley.

3.Denisov, A.M., and Lorenzi, A., 1992, «Identification of Nonlinear Terms in Boundary Value Problems Related to Ordinary Differential Equations», Differential and Integral Equations, Vol.5, pp.567-579.

4.Denisov, A.M., and Solov’yeva, S.I., 1993, «The Problem of Determining the Coefficient in the Non-Linear Stationary Heat Conduction Equation», Comp. Maths. Math. Phys., Vol.33, pp. 1145-1153.

5.Kamimura, Y., 1998, «An Inverse Problem of Determining a Nonlinear Term in an Ordinary Differential Equation» Vol.11, pp.341-359.

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