Sequential Function Estimation - "Beck's Method"

Keith A. Woodbury

Mechanical Engineering Department

The University of Alabama

Overview

- Function Estimation vs. Parameter Estimation
- Function *Specification* - reduces function to collection of parameters
- Ill-posedness: Future times regularization
- Becks_method.exe - simple program

Parameter Estimation

Given a system model with one or more unknown parameters, adjust these parameters so that the model reproduces the observed behavior of the system.

Parameter Estimation Example

- Simple heat conduction

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

$$\left.\frac{\partial T}{\partial t}\right|_{x=L} = 0$$

$$-k \left.\frac{\partial T}{\partial t}\right|_{x=0} = q_0 = const$$

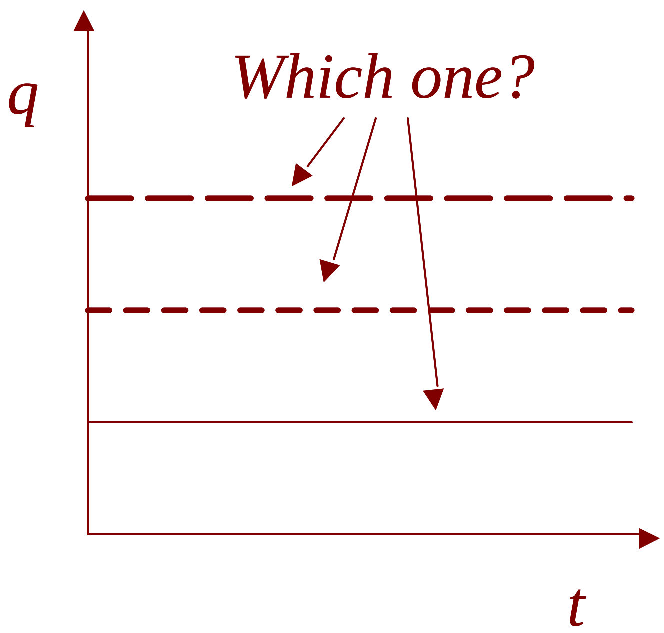
$$T(x,0) = T_0 = const$$

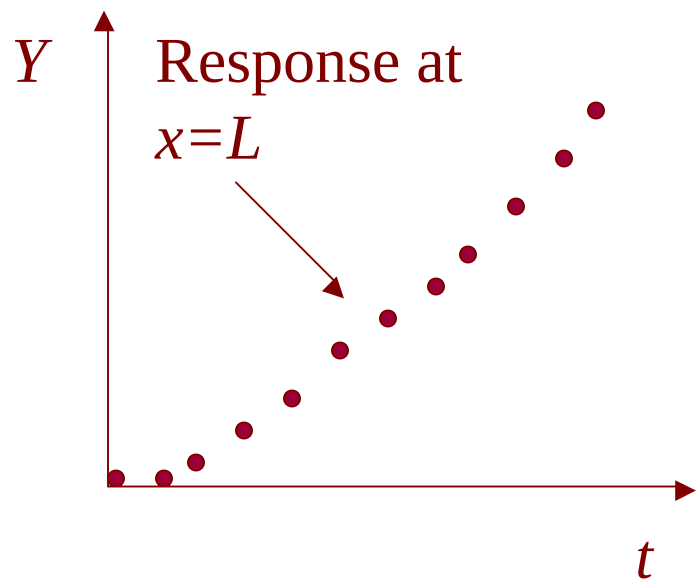
Either α or q_0 could be unknown.

A suitable observation of the system's behavior (set of data) must be available

Parameter Estimation Example

- Suppose q_0 is unknown





Parameter Estimation Example

- "Match" the observed data *Y* with the model-computed values *T* in a least squares sense
- Minimize $$\boxed{S = \sum_{i=1}^{n} \left(Y_i - T_i\right)^2 = \left(\mathbf{Y} - \mathbf{T}\right)^T \left(\mathbf{Y} - \mathbf{T}\right)}$$
- Note the solution $T(x,t)$ for constant q_o is

$$T(x,t) = \frac{q_0 L}{k}\left[\frac{\alpha t}{L^2} + \frac{1}{3} - \frac{x}{L} + \frac{1}{2}\left(\frac{x}{L}\right)^2 - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\exp\left(-\frac{n^2\pi^2\alpha t}{L^2}\right)\cos\left(\frac{n\pi x}{L}\right)\right] + T_0$$

Parameter Estimation Example

- To minimize the sum squared error S, force the derivative of S with respect to q_0 to zero

$$\frac{dS}{dq_0} = 2\sum_{i=1}^{n}(Y_i - T_i)\frac{\partial T_i}{\partial q_0} = 0$$

$$= \sum_{i=1}^{n}(Y_i - T_i)X_{q_i}$$

Sensitivity of T_i with respect to parameter q_0

Parameter Estimation Example

- Note the solution *T(x,t)* for constant q_o is

$$T(x,t)=\frac{q_0L}{k}\left[\frac{\alpha t}{L^2}+\frac{1}{3}-\frac{x}{L}+\frac{1}{2}\left(\frac{x}{L}\right)^2-\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\exp\left(-\frac{n^2\pi^2\alpha t}{L^2}\right)\cos\left(\frac{n\pi x}{L}\right)\right]+T_0$$

which can be written compactly as

$$T(x,t)=\frac{q_0L}{k}\left[f_1(t)+f_2(x)+g(x,t)\right]+T_0$$

so the sensitivity function is

$$X_q(x,t)\equiv\frac{\partial T}{\partial q_0}=\frac{L}{k}\left[f_1(t)+f_2(x)+g(x,t)\right]$$

Parameter Estimation Example

- Substituting

$$\sum_{i=1}^{n}\left(Y_i - T_i\right)X_{qi} = 0$$

$$\sum_{i=1}^{n}\left(Y_i - \left(\frac{q_0 L}{k}\left[f_1(t_i) + f_2(x^*) + g(x^*, t_i)\right] + T_0\right)\right) \times$$

$$\frac{L}{k}\left[f_1(t_i) + f_2(x^*) + g(x^*, t_i)\right] = 0$$

Parameter Estimation Example

- The circled terms are equal to X_{qi}

$$\sum_{i=1}^{n}\left(Y_i - T_i\right)X_{qi} = 0$$

$$\sum_{i=1}^{n}\left(Y_i - \left(\frac{q_0 L}{k}\left[f_1(t_i) + f_2(x^*) + g(x^*, t_i)\right] + T_0\right)\right) \times$$

$$\frac{L}{k}\left[f_1(t_i) + f_2(x^*) + g(x^*, t_i)\right] = 0$$

Parameter Estimation Example

- So

$$\sum_{i=1}^{n} \left(Y_i - q_0 X_{qi} + T_0 \right) X_{qi} = 0$$

which can be solved for q_0!

$$\boxed{q_0 = \frac{\sum_{i=1}^{n} \left(Y_i - T_0 \right) X_{qi}}{\sum_{i=1}^{n} X_{qi} X_{qi}} = \left[\mathbf{X_q}^T \mathbf{X_q} \right]^{-1} \mathbf{X_q}^T \left(\mathbf{Y} - \mathbf{T_0} \right)}$$

Function Estimation

- How does function estimation differ from parameter estimation?

  Instead of a single (or finite) group of unknowns, there is a continuous function to be determined.

Function Estimation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

$$\left.\frac{\partial T}{\partial t}\right|_{x=L} = 0$$

$$-k\left.\frac{\partial T}{\partial t}\right|_{x=0} = q(t) \neq const$$

$$T(x,0) = T_0 = const$$

Problem 1

Function Estimation

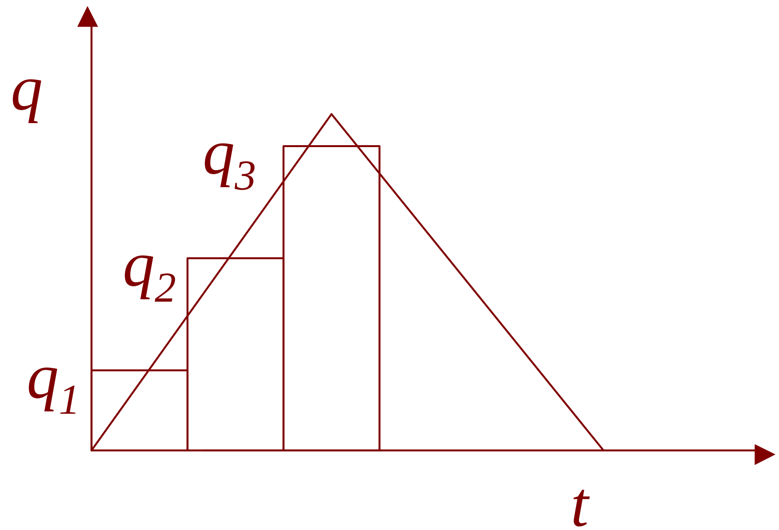
- Basic Idea: *Reduce the continuous function to a set of parameters by specifying an underlying nature of the function.* Possibilities:

  constant, linear, quadratic, etc - these are all *global* or *whole domain* functions

  piecewise constant, piecewise linear, etc - these are all *local* or *sequential* functions

Function Estimation Approach

- Use superposition for this linear problem to transform the original problem into a sequence of parameter estimation problems



Piecewise constant approximation

Duhamel's Summation

- Denote the solution to Problem 1 with $q(t)=1$ and $T_0=0$ as $j\,(x,t)$.
- Then the solution to Problem 1 with $q(t)=q_1=$const is

$$T(x,t)=q_1\varphi(x,t)$$

- and the solution for a pulse of magnitude q_1 of duration Dt is

$$T(x,t)=q_1\big(\varphi(x,t)-\varphi(x,t-\Delta t)\big)$$

Duhamel's Summation

- The solution to Problem 1, up to some time t_M, with $q(t)$ as a sequence of pulses of width Dt is the sum of the responses for all the pulses:

$$
\begin{aligned}
T(x,t_M) = {} & q_1\big(\varphi(x,t_M) - \varphi(x,t_M - \Delta t)\big) \\
& + q_2\big(\varphi(x,t_M - \Delta t) - \varphi(x,t_M - 2\Delta t)\big) \\
& + q_3\big(\varphi(x,t_M - 2\Delta t) - \varphi(x,t_M - 3\Delta t)\big) \\
& \ \vdots \\
& + q_M\big(\varphi(x,t_M - (M-1)\Delta t) - \varphi(x,t_M - M\Delta t)\big)
\end{aligned}
$$

Duhamel's Summation

- If the initial temperature is T_0, then

$$T(x,t_M) = T_0 + \sum_{n=1}^{M} q_n \left(\varphi(x, t_M - (n-1)\Delta t) - \varphi(x, t_M - n\Delta t) \right)$$

$$= T_0 + \sum_{n=1}^{M} q_n \left(\varphi(x, t_{M-n+1}) - \varphi(x, t_{M-n}) \right)$$

$$\boxed{= T_0 + \sum_{n=1}^{M} q_n \Delta\varphi_{M-n}}$$

Where $\Delta\varphi_i \equiv \varphi(x, t_{i+1}) - \varphi(x, t_i)$

Duhamel's Summation

- Matrix form

$$
\begin{Bmatrix} T_1 \\ T_2 \\ \vdots \\ T_M \\ \vdots \\ T_{M+r-1} \end{Bmatrix}
=
\begin{bmatrix}
\Delta\varphi_0 & & & & & \\
\Delta\varphi_1 & \Delta\varphi_0 & & & & \\
\vdots & \vdots & \ddots & & & \\
\Delta\varphi_{M-1} & \cdots & \Delta\varphi_1 & \Delta\varphi_0 & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & \\
\Delta\varphi_{M+r-2} & \Delta\varphi_{M+r-3} & \Delta\varphi_{M+r-4} & \cdots & \Delta\varphi_1 & \Delta\varphi_0
\end{bmatrix}
\begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_M \\ \vdots \\ q_{M+r-1} \end{Bmatrix}
$$

$$
+ \begin{bmatrix} T_0 & T_0 & \cdots & T_0 & \cdots & T_0 \end{bmatrix}^T
$$

Stoltz Method

- In matrix notation

$$\{\mathbf{T}\} = [\mathbf{X}]\{\mathbf{q}\} + \{\mathbf{T_0}\}$$

- If the calculated values **T** are matched exactly to the observations **Y**, then **q** could be solved directly $\{\mathbf{q}\} = [\mathbf{X}]^{-1}\{\mathbf{Y} - \mathbf{T_0}\}$

  or premultiply by $\mathbf{X}^T$ and solve for **q**

$$\{\mathbf{q}\} = \left[\mathbf{X}^T\mathbf{X}\right]^{-1}\mathbf{X}^T\{\mathbf{Y} - \mathbf{T_0}\}$$

Stoltz Method

- This "exacting matching" of data is sometimes referred to as the Stoltz method
  - can be applied either as *whole domain* method (as suggested above) or sequentially
  - the inherent *ill-posedness* of this problem is manifested by the poor condition number of the coefficient matrix $\mathbf{X}^T\mathbf{X}$
    - ill-posedness gives large changes in output **q** for small changes in the input **Y** or small values of Δt

Sequential Stoltz Method

- From the equation for T_M

$$q_M = \frac{\left(Y_M - \sum_{i=1}^{M-1} q_i \Delta\varphi_{M-i} - T_0\right)}{\Delta\varphi_0}$$

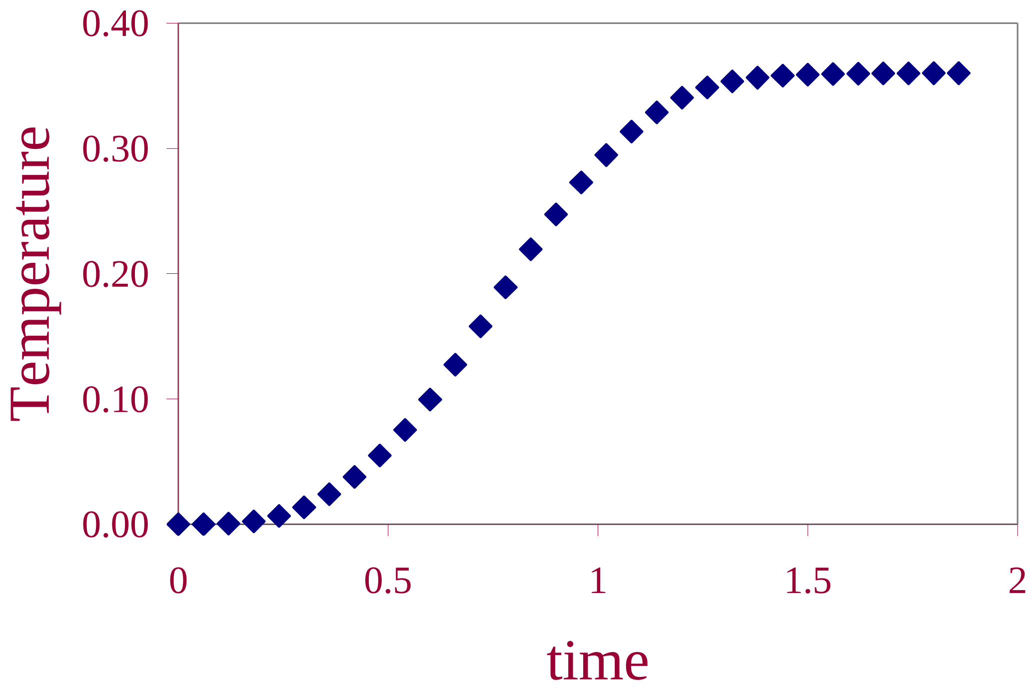
$$= \frac{\left(Y_M - \sum_{i=1}^{M-1} q_i \Delta\varphi_{M-i} - T_0\right)}{(\varphi_1 - \varphi_0)}$$

Stoltz's Method

$$= \frac{\left(Y_M - \sum_{i=1}^{M-1} q_i \Delta\varphi_{M-i} - T_0\right)}{\varphi_1}$$

Stoltz Method: Example

- Material with $k=\rho c_p=1$ with $T_0=0$ and data from $x=L=1$

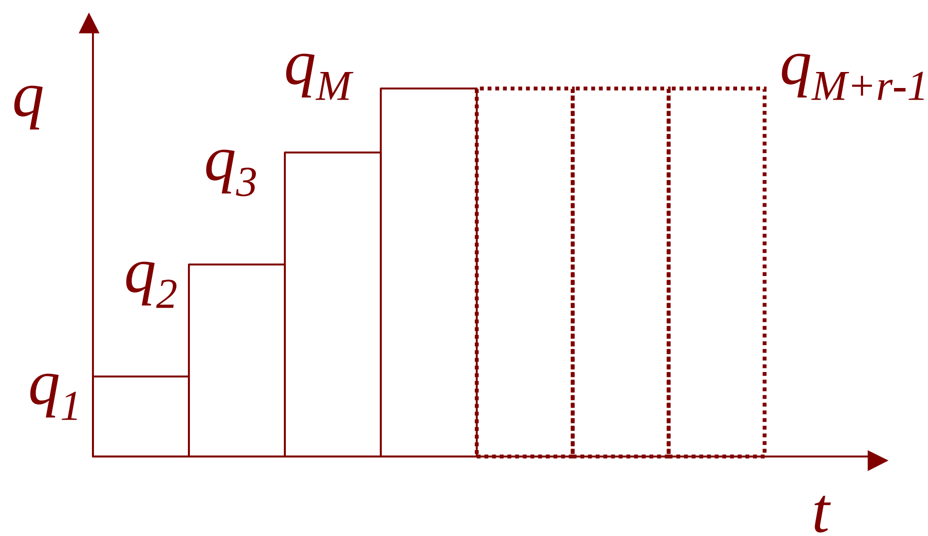


Stoltz Method: Results

time	flux	time	flux
0.03	3.10E-02	0.93	-4.39E+14
0.09	8.00E-02	0.99	6.80E+15
0.15	2.99E-01	1.05	-1.05E+17
0.21	-2.10E+00	1.11	1.63E+18
0.27	3.61E+01	1.17	-2.53E+19
0.33	-5.54E+02	1.23	3.91E+20
0.39	8.59E+03	1.29	-6.06E+21
0.45	-1.33E+05	1.35	9.38E+22
0.51	2.06E+06	1.41	-1.45E+24
0.57	-3.19E+07	1.47	2.25E+25
0.63	4.94E+08	1.53	-3.48E+26
0.69	-7.64E+09	1.59	5.39E+27
0.75	1.18E+11	1.65	-8.35E+28
0.81	-1.83E+12	1.71	1.29E+30
0.87	2.84E+13	1.77	-2.00E+31
		1.83	3.10E+32

Beck's Sequential Method

- Add stability to exact matching by considering r future time steps and making the temporary (but incorrect) assumption that $q_M = q_{M+1} = \ldots = q_{M+r-1}$



Beck's Sequential Method

- Write the equations for T_M, T_{M+1}, …, T_{M+r-1} from the Duhamel's summation

$$T_M = \sum_{i=1}^{M-1} q_i \Delta\varphi_{M-i} + T_0 + \varphi_1 q_M$$

$$T_{M+1} = \sum_{i=1}^{M-1} q_i \Delta\varphi_{M-i+1} + T_0 + \varphi_2 q_M$$

$$\vdots \quad = \quad \vdots$$

$$T_{M+r-1} = \sum_{i=1}^{M-1} q_i \Delta\varphi_{M-i+r-1} + T_0 + \varphi_r q_M$$

Beck's Sequential Estimation

- Now use least-squares to match the data in an average sense. Minimize

$$S = \sum_{i=1}^{r} \left(Y_{M+i-1} - T_{M+i-1} \right)^2$$

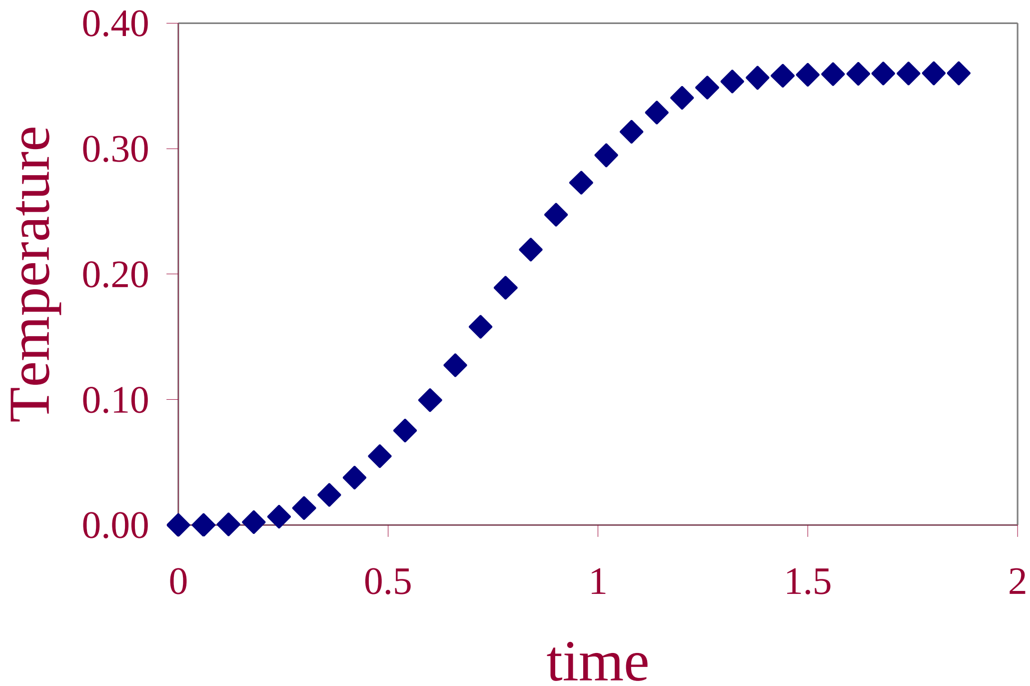
- This leads to

Beck's Method

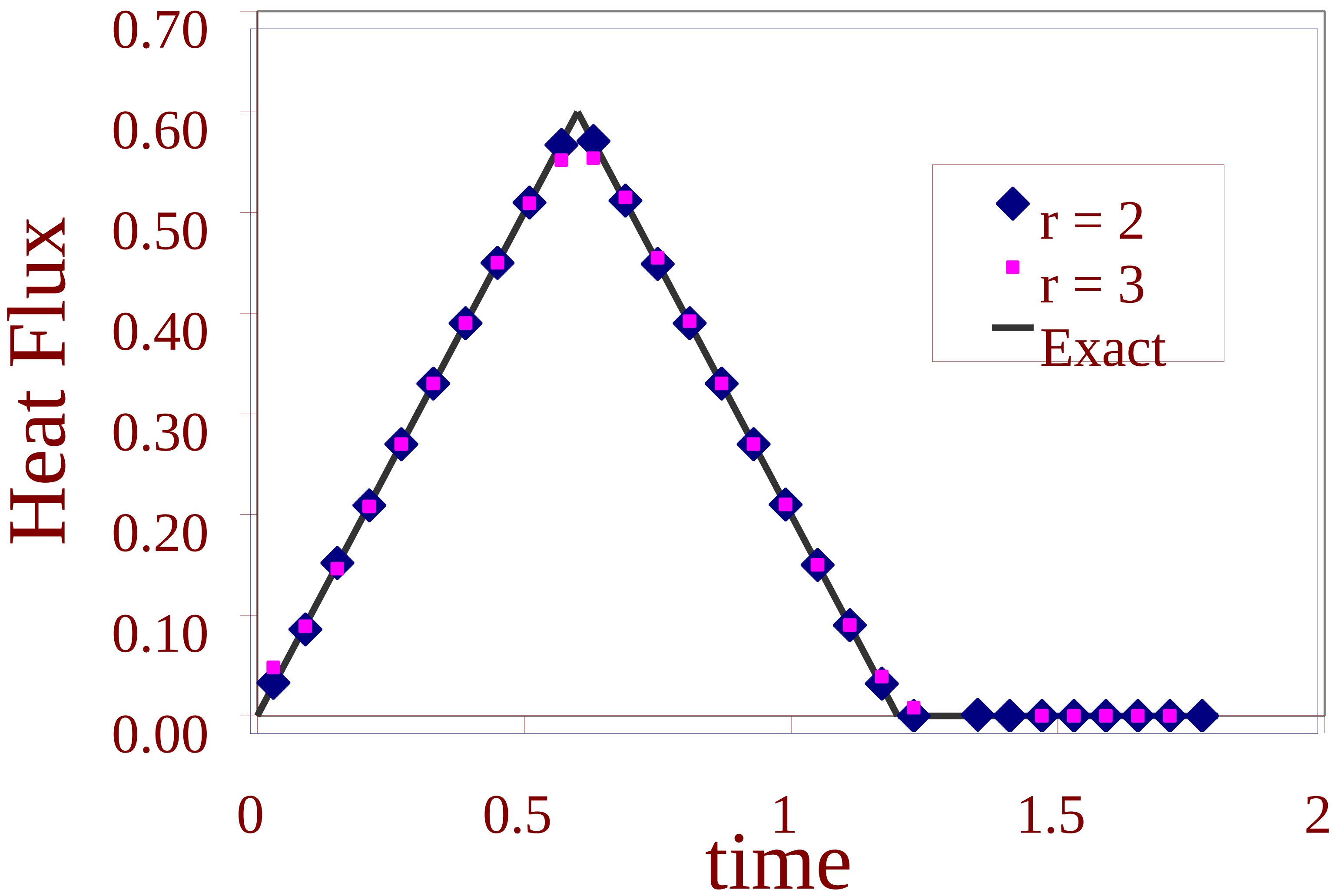
$$\boxed{q_M = \frac{\sum_{i=1}^{r} \left(Y_{M+i-1} - \sum_{k=1}^{M-1} q_k \Delta\varphi_{M-k+i-1} - T_0 \right) \varphi_i}{\sum_{i=1}^{r} \varphi_i^{\,2}}}$$

Beck’s Method Example

- Material with $k=\rho c_p=1$ with $T_0=0$ and data from $x=L=1$ (same as before)



Beck's Method Results



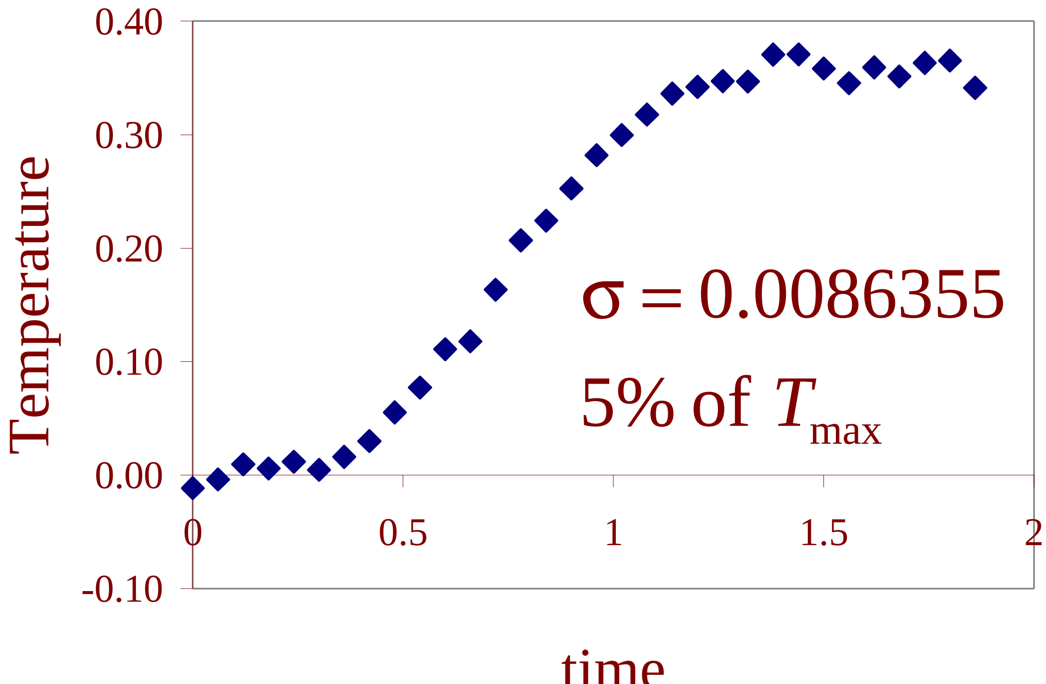
Noise in Data

- All data gathered experimentally contains some amount of error (or noise) associated with the measurement process
- If the noise is random and normally distributed with zero mean and standard deviation s, then an error over the range of +/– 1.96 s will encompass 95% of the data

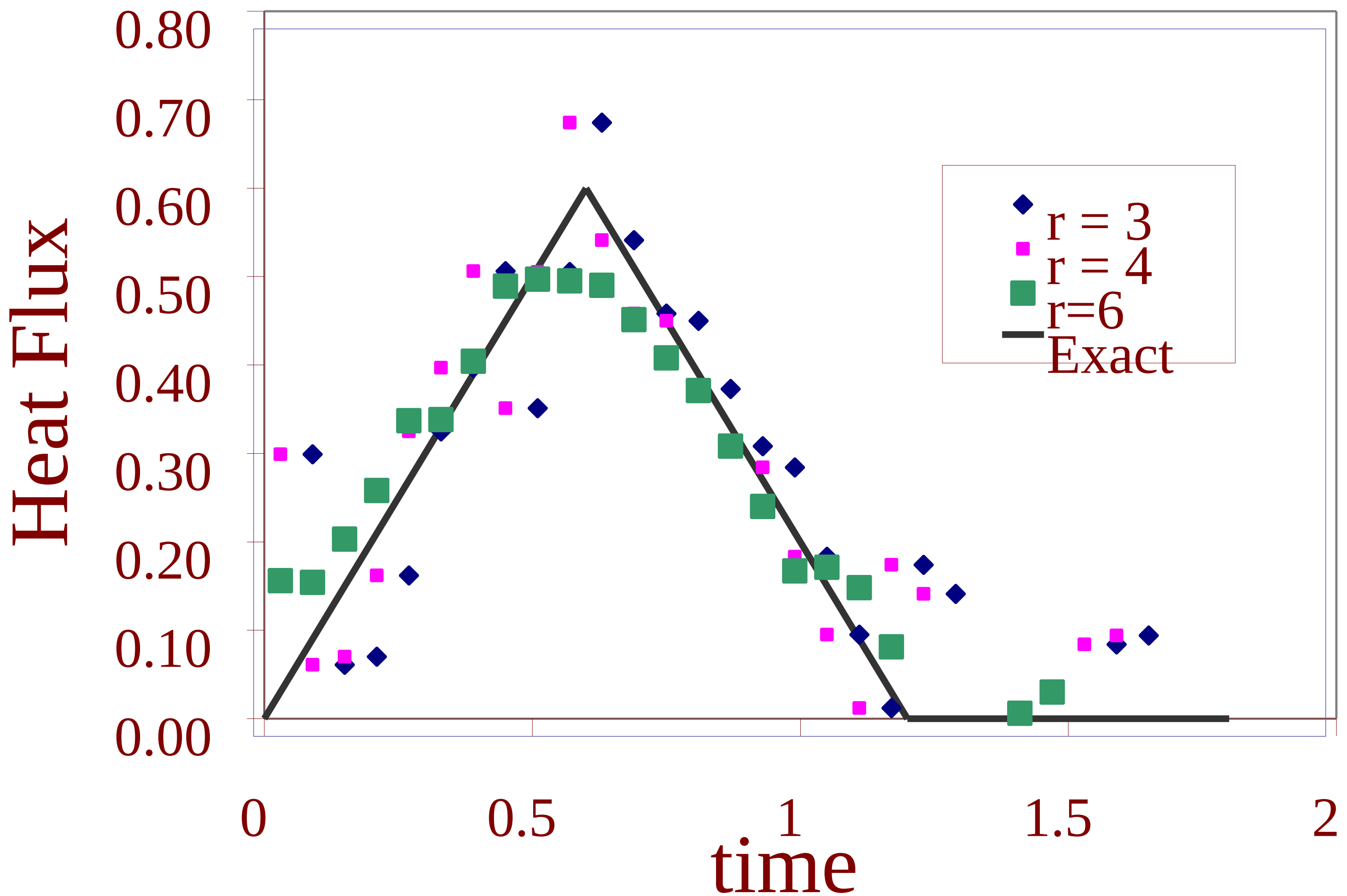
$$\pm \Delta T = 1.96\sigma$$

Noise in Data

- To test the ability of the algorithm to perform in the face of noise, we "poison" the data by adding random perturbations from a normal distribution to the data



Noised Results



Regularization

- The use of future times information provides a stabilization, or *regularization*, of the underlying ill-posed problem
  – concurrently, the use of future times information induces *bias* in the estimate for q_i

Becks_Method.exe

- Windows Console Application (runs from command line)
  - Two input files
    - thermophysical property data
    - temperature data
  - keyboard input required
    - define which sensors to consider and how many future times to include
- “C” source code included
  - Closely follows algorithm in Beck, et al. (1985)

Becks_Method.exe

- Thermophysical Property data file
  - “therm.nod”

```
1          /* total number of sensors  */
1              /* indicies of the sensors  */
1.             /* length of 1-d domain */
1.    1.       /* k, rho*cp for AL 5083 alloy  */
1.             /* number of distinct TCs locations  */
1.             /* dimensional locations of the TCs */
```

Becks_Method.exe

- Second thermophysical property file example
  - "therm.al"

```
21              /* total number of sensor  */
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
1.6875          /* length of 1-d domain in inches */
1.6e-3  2.06e-2  /* k, rho*cp for AL 5083 alloy  */
1               /* number of distinct TCs locations  */
1.6875          /* dimensional locations of the TCs */
```

Becks_Method.exe

- Temperature data file
  - "triangle.dat"

times	*temperatures*
0.000	0.000000
0.060	0.000024
0.120	0.000494
0.180	0.002368
0.240	0.006564
0.300	0.013648
0.360	0.023937
.	.
.	.
.	.

Becks_Method.exe

- Command Line execution

    C:> Becks_Method therm.nod triangle.dat

- redirection will capture output to a file

    C:> Becks_Method therm.nod triangle.dat > beck.out

References

- Beck, Blackwell, and St. Clair, *Inverse Heat Conduction: Ill-Posed Problems*, John-Wiley, 1985 (out of print; copies available from J.V. Beck)