Regularized Inversion of Real-Valued Laplace Transforms.

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Abstract. The paper presents one-parametric set of regularizing operators of convolution integral type for inverting of real-valued Laplace transforms. Provided error analysis shows advantages and restrictions of the proposed method and reflects some general limitations for any method of real-valued Laplace transforms inversion.

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1. Introduction

Laplace transformation is used in solving different problems arising from various spheres of science and engineering. Simple image functions can be treated with the help of transform tables or by analytical evaluation of Bromwich contour integral. In case when Laplace transform has been obtained numerically or from an experiment a numerical inversion becomes the only way to find required solution. In many cases a Laplace transform is known on the real axis only.

It is a well-known fact that numerical inversion of integral transforms cannot be based on exact formulas of inverse transformation because of the problem ill-posedness [16], and, therefore, Tikhonov regularization should be applied in order to obtain stable solution. Regularizing operators for integral equations of convolution type [16] solve the problem when real formula of inverse transformation is given. The problem becomes more complicated when inverse transformation is given in form of Bromwich contour integral, whereas given integral transform is known on the real positive axis [17, 18].

There is a number of methods that use regularization (see, for example, [1]–[4], [6], [7]). However, in all known regularization methods of Laplace inverse transformation the regularization step follows by other transformation such as discretization or decomposition into a series of orthogonal polynomials. That is regularization is applied to a second-order problem, not to a problem of inverting of real-valued Laplace transforms itself. In this case it turns out to be impossible to determine restrictions and limitations of a proposed method differently than by caring out an actual implementation of the method on a subset of Laplace transforms.

In case of real-valued Laplace transforms effective solution of the problem has been obtained in [10]. Then results have been generalized by introducing an integer parameter [11].

In this paper further generalization is provided by building one-parametric set of regularizing operators. Analytical link between regularized and exact inverse transforms allows to analyze general features and limitations of regularized numerical inversion of Laplace transforms. It gives information about the rate of convergence, that could be useful while solving particular problems. The applicability of given method was illustrated by various examples [10]. Determination of regularization parameter is briefly discussed in [11].

2. Outline of the inversion method

The Laplace transform of a function $f(t)$ is:

$$F(p) = \int_0^\infty \mathrm{e}^{-pt} f(t)\,\mathrm{d}t \ . \tag{1}$$

Considering (1) as integral equation with respect to $f(t)$ it is shown in [10] that in case when a Laplace transform $F(p)$ is given for real $p > 0$, its regularized inverse Laplace transform $f_R(t)$ can be found by calculating the following convolution integral:

$$f_R(t) = \int_0^\infty F(u)\Pi(R, tu)\,\mathrm{d}u \ . \tag{2}$$

The kernel of inverse Laplace transformation $\Pi(R, x)$ is given by formula:

$$\Pi(R,x) = \frac{-2\cosh \pi R}{\pi^2 \tanh \pi R}\,\mathrm{Im}\left[\Gamma\left(iR - \tfrac{1}{2}\right) x^{\frac{1}{2}-iR}\,{}_1F_2\left(1; \frac{5}{4} - \frac{iR}{2}, \frac{3}{4} - \frac{iR}{2}, \frac{x^2}{4}\right)\right], \tag{3}$$

where $\Gamma(z)$ is gamma function, ${}_1F_2(1;a,b;x)$ is a generalized hypergeometric function, parameter $R=\pi^{-1}\cosh^{-1}(\alpha^{-1})$ depends only on parameter of regularization α.
It is also shown in [10] that regularized inverse Laplace transform $f_R(t)$ is connected to the exact one $f(t)$ as follows:

$$f_R(t)=\frac{2\coth\pi R}{\pi}\int_0^\infty f(tu)\sqrt{u}\frac{\sin(R\ln u)}{u^2-1}\,\mathrm{d}u. \tag{4}$$

Let us define a connection between regularized and exact inverse integral transforms in form of equation (4):

$$f_R(t)=\frac{2}{\pi}k(R,a)\int_0^\infty f(tu)u^a\frac{\sin(R\ln u)}{u^2-1}\,\mathrm{d}u, \tag{5}$$

where a is a real parameter, and coefficient $k(R,a)\to 1$ as $R\to\infty$.
It is apparent that integral (5) is absolutely convergent for every $t>0$ if $f(t)$ is of bounded variation and the following conditions are satisfied:

$$f(t)t^{a+1}=o(t^\epsilon),\ \epsilon>0,\ \text{as } t\to 0, \tag{6}$$

$$f(t)t^{a-1}=o(t^{-\epsilon}),\ \epsilon>0,\ \text{as } t\to \infty. \tag{7}$$

Theorem 1 *If function f(t) is piecewise continuous, has piecewise continuous derivative, and satisfies conditions (6), (7) then for every* $t>0$

$$\lim_{R\to\infty} f_R(t)=\frac{f(t+)+f(t-)}{2}. \tag{8}$$

The proof of this theorem is similar to the one provided for Theorem 1 in [10].

Let $F(p)$, $f(t)$ and $F_R(p)$, $f_R(t)$ be Laplace-transform pairs. Applying Laplace transformation to (5) and inverting the order of integration we get:

$$F_R(p)=\frac{2}{\pi}k(R,a)\int_0^\infty F(px)\frac{x^{1-a}}{x^2-1}\sin(R\ln x)\,\mathrm{d}x\ . \tag{9}$$

Integral (9) converges absolutely for every real $p>0$ under the following conditions:

$$F(p)p^{2-a}=o(p^\epsilon),\ \epsilon>0,\ \text{as } p\to 0, \tag{10}$$

$$F(p)p^{-a}=o(p^{-\epsilon}),\ \epsilon>0,\ \text{as } p\to \infty. \tag{11}$$

Equation (9) is of type (5), therefore $F_R(p)\to F(p)$ as $R\to\infty$ for every real $p>0$. For complex parameter p, integral (9) is convergent and $F_R(p)\to F(p)$ if Laplace transform $F(p)$ satisfies conditions (10), (11) and is analytic for $\mathrm{Re}\,p\geq 0$ except for $p=0$, or may have isolated singular points on the imaginary axis of type $1/(p\pm ib)^r$, where $0<r<1$, $b>0$.

Next step is to find regularized solution of equation (1) in form of (2). Applying Laplace transformation to equation (2) we get:

$$F_R(p)=\int_0^\infty F(pu)\,\mathrm{d}u\int_0^\infty \mathrm{e}^{-x}\Pi(R,ux)\,\mathrm{d}x\ . \tag{12}$$

Equations (9) and (12) are identical if there is a function $\Pi(R,x)$ such that

$$\int_0^\infty \mathrm{e}^{-x}\Pi(R,ux)\,\mathrm{d}x=\frac{2}{\pi}k(R,a)\frac{u^{1-a}}{u^2-1}\sin(R\ln u)\ . \tag{13}$$

The last equation is the integral equation of the first kind of the convolution type with respect to function $\Pi(R,x)$. Equation (13) can be solved by means of Mellin transformation. Indeed, applying Mellin transformation to (13) we get:

$$\Gamma(1-s)\mathcal{M}[\Pi(R,x);s] = \mathcal{M}\left[\frac{2}{\pi}k(R,a)\frac{x^{1-a}}{x^2-1}\sin(R\ln x);s\right] \quad , \qquad (14)$$

where $\mathcal{M}$, s are Mellin transformation operator and parameter correspondently.
The Mellin transform in right-hand side of equation (14) can be obtained directly using standard integrals [15]:

$$\mathcal{M}\left[\frac{2}{\pi}\frac{x^{1-a}}{x^2-1}\sin(R\ln x);s\right] = \frac{\sinh\pi R}{\cosh\pi R+\cos\pi(s-a)}. \qquad (15)$$

Taking into account that $\Gamma(s)\Gamma(1-s) = \pi/\sin\pi s$ [13], Mellin transform of function in question can be presented as follows:

$$\mathcal{M}[\Pi(R,x);s] = \Pi(R,s) = \frac{k(R,a)}{\pi}\Gamma(s)\frac{\sinh\pi R}{\cosh\pi R+\cos\pi(s-a)} \ . \qquad (16)$$

In equation (16) the factor $\sinh\pi R/(\cosh\pi R + \cos\pi(s-a))$ plays the role of a stabilizing factor [10, 11]. Equation (16) is the basic equation for finding regularizing operators. If equation (16) is solvable, then there is a function $\Pi(R,x)$ for which equations (9) and (12) are identical. Therefore, because of uniqueness of Laplace transforms, equations (2) and (5) will represent one and the same function $f_R(t)$.

Note that considered method is not specific to Laplace transformation. Whenever it is possible to compose and solve equation (16), this method could be used for other integral transformations of Mellin convolution type [12].

In order to invert equation (16) let us rewrite (16) in a "more symmetrical" form:

$$\Pi(R,s) = \frac{2k(R,a)}{\pi}\Gamma(s)\sin\frac{\pi s}{2}\sinh\pi R\,\frac{\cos\frac{\pi}{2}(s-a)\cos\frac{\pi}{2}a - \sin\frac{\pi}{2}(s-a)\sin\frac{\pi}{2}a}{\cosh\pi R+\cos\pi(s-a)}. \qquad (17)$$

Taking into account the following identities [9]: §

$$\mathcal{M}[\sin x;s] = \Gamma(s)\sin\frac{\pi s}{2}, \qquad (18)$$

$$\mathcal{M}\left[\frac{x^a\cos(R\ln x)}{x^2+1};1-s\right] = \frac{\pi\cosh\frac{\pi R}{2}\cos\frac{\pi}{2}(s-a)}{\cosh\pi R+\cos\pi(s-a)} \quad , \qquad (19)$$

$$\mathcal{M}\left[\frac{x^a\sin(R\ln x)}{x^2+1};1-s\right] = \frac{-\pi\sinh\frac{\pi R}{2}\sin\frac{\pi}{2}(s-a)}{\cosh\pi R+\cos\pi(s-a)} \quad , \qquad (20)$$

inverse Mellin transform of $\Pi(R,s)$ in (17) can be obtained by calculating the following integrals:

$$\Pi(R,x) = \frac{4k(R,a)}{\pi^2}\left[\sinh\frac{\pi R}{2}\cos\frac{\pi a}{2}\int_0^\infty \frac{u^a\cos(R\ln u)}{u^2+1}\sin xu\,\mathrm{d}u + \right.$$
$$\left. + \ \cosh\frac{\pi R}{2}\sin\frac{\pi a}{2}\int_0^\infty \frac{u^a\sin(R\ln u)}{u^2+1}\sin xu\,\mathrm{d}u\right] \ .$$

§ Equations (19), (20) can be verified by direct calculations of Mellin transforms with the help of standard integrals [15]

Applying standard integrals [15] we will have:

$$\Pi(R,x) = \frac{-2k(R,a)}{\pi^2} \times \tag{21}$$
$$\times \mathrm{Im}\left[\sin\pi(a+iR)\Gamma(a-1+iR)x^{1-a-iR}{}_1F_2\left(1; \frac{2-a-iR}{2}, \frac{3-a-iR}{2}; \frac{x^2}{4}\right)\right] ,$$

where according to convergence conditions of standard integrals $-2 < a < 2$.
The convergence conditions of integral (2) follow from analysis of continuous function $\Pi(R,x)$. In accordance with definition [8], function ${}_1F_2\left(1; \frac{2-a-iR}{2}, \frac{3-a-iR}{2}, \frac{-x^2}{4}\right) \to 1$ as $x \to 0$. Therefore

$$\Pi(R,x) = O(x^{1-a}) \text{ as } x \to 0, \tag{22}$$

Representing function ${}_1F_2(1; \frac{2-a-iR}{2}, \frac{3-a-iR}{2}, \frac{-x^2}{4})$ in terms of confluent hypergeometric functions and using asymptotic expansion of the latter [8, 13] it can be shown that

$$\mid \Pi(R,x) \mid = O(x^{-1-a}) \text{ as } x \to \infty. \tag{23}$$

It follows from (22), (23) that integral (2) converges under conditions (10), (11).

Let us analyze equation (21) in more details. Because of factor $x^{-iR} = \cos(R\ln x) - i\sin(R\ln x)$, equation (21) can be written in an equivalent form:

$$\Pi(R,x) = A(R,x)\cos(R\ln x) + B(R,x)\sin(R\ln x). \tag{24}$$

It is apparent from (21) that $\lim_{R\to\infty}\Pi(R,x)$ does not exist for all $x \neq 0$, because $A(R,x) \to \infty$, and $B(R,x) \to \infty$ as "frequency" $R \to \infty$. In other words, real transfer function of inverse Laplace transformation does not exist.

3. Proof of regularization

It has been proven in Theorem 1 that $f_R(t) \to f(t)$ as $R \to \infty$ at any $t > 0$ where $f(t)$ is continuous. Let $\mathcal{U}^{(a)}$ denote a subspace of pre-image functions that satisfy conditions of Theorem 1 for some fixed, valid value of parameter a. Let $\mathcal{F}^{(a)}$ denote the corresponding subspace of Laplace transforms transforms, which satisfy convergence conditions for integral (2).

It is apparent that integral (2) is convergent if the following integral exists:

$$\| F \| = \int_0^\infty w(x) \mid F(x) \mid \mathrm{d}x, \tag{25}$$

where $w(x) = x^{1-a}/(1+x^2)$, $-2 < a < 2$.
As it follows from (21), function $\Pi(R,x)$ is continuous for $x > 0$. Analyzing the weight function $w(x)$ and conditions (22) , (23) we state that limits $\lim_{x\to 0}\Pi(R,x)/w(x)$, and $\lim_{x\to\infty}\Pi(R,x)/w(x)$ exist. Therefore

$$\mid \Pi(R,x) \mid \leq A(R)w(x). \tag{26}$$

Let $\mathcal{L}_R^{-1}$ denote operator defined by equations (2), (21) for some fixed $-2 < a < 2$. If $F_1(p) \in \mathcal{F}^{(a)}$, $F_2(p) \in \mathcal{F}^{(a)}$ and

$$\| F_1 - F_2 \| = \int_0^\infty w(x) \mid F_1(x) - F_2(x) \mid \mathrm{d}x < \delta, \tag{27}$$

then we can prove the continuity of the operator $\mathcal{L}_R^{-1}$.

Theorem 2 *Operator $\mathcal{L}_R^{-1}$ from $\mathcal{F}^{(a)}$ to $\mathcal{U}^{(a)}$ is continuous with respect to $F(p)$.*

Proof. Estimating the difference $\Delta\mathcal{L}_R^{-1}$:

$$\Delta\mathcal{L}_R^{-1} = \mathcal{L}_R^{-1}[F_1(p)] - \mathcal{L}_R^{-1}[F_2(p)] = \int_0^\infty \Pi(R, tx)[F_1(x) - F_2(x)]\,\mathrm{d}x \quad , (28)$$

and taking into account (26), (27) we get:

$$|\Delta\mathcal{M}_R^{-1}| \leq \int_0^\infty A(R,t)w(x)|F_1(x) - F_2(x)|\,\mathrm{d}x \leq A(R,t)\delta. \tag{29}$$

The continuity of the operator $\mathcal{L}_R^{-1}$ follows.

Thus we have proved that $\mathcal{L}_R^{-1}$ is a continuous operator with respect to $F(p)$ and $\lim_{R\to\infty} f_R(t) = f(t)$ for any t where $f(t)$ is continuous. Therefore, the following theorem is a corollary of Tikhonov theorem [10, p.49].

Theorem 3 *For any fixed $-2 < a < 2$ operators $\mathcal{L}_R^{-1}$ from $\mathcal{F}^{(a)}$ to $\mathcal{U}^{(a)}$ are regularizing operators of inverse Laplace transformation.*

It follows from the regularization theory that each of built regularizing operators defines a stable method for finding inverse Laplace transforms.

4. Error analysis

The fact that a regularized solution tends to the exact one when $R \to \infty$, i.e. under unbounded increase of input data accuracy, provides no information about the rate of convergence. We may assume that the rate of convergence depends on location and type of Laplace transform singular points. Equation (5) allows to investigate errors analytically as well as the rate of convergence similarly to the one provided in [10] for $a = 1/2$, and in [11] for $a = 1/2 - k$.

Consider the following Laplace-transform pair:

$$F(p) = \frac{1}{(p+z)^r}, \quad f(t) = \frac{1}{\Gamma(r)} t^{r-1} \exp(-zt), \quad r > 0,\ \mathrm{Re} z \geq 0 \ . \tag{30}$$

In this case (5) will become:

$$f_R(t) = \frac{2k(R,a)}{\pi} \frac{t^{r-1}}{\Gamma(r)} \int_0^\infty u^{a+r-1} \exp(-ztu) \frac{\sin(R \ln u)}{u^2 - 1}\,\mathrm{d}u \ . \tag{31}$$

Integral (31) converges on the lower limit if $a + r > 0$. In case of $\mathrm{Re}\, z = 0$ we have an additional condition: $a + r < 2$.

Evaluating integral (31) we get:

$$\begin{aligned} f_R(t) = \frac{t^{r-1}}{\Gamma(r)} k(R,a) \Bigg\{ & \exp(-zt) \frac{\sinh \pi R \cosh \pi R}{\cosh^2 \pi R - \cos^2 \pi A} + \\ + \ & \exp(zt) \frac{\sinh \pi R \cos \pi A}{\cosh^2 \pi R - \cos^2 \pi A} + \\ + \ & \frac{1}{\pi i} \Bigg[\Gamma(A - 2 + iR)(zt)^{2-A-iR} {}_1F_2\left(1; \frac{4 - A - iR}{2}, \frac{3 - A - iR}{2}, \frac{z^2t^2}{4}\right) - \\ & - \Gamma(A - 2 - iR)(zt)^{2-A+iR} {}_1F_2\left(1; \frac{4 - A + iR}{2}, \frac{3 - A + iR}{2}, \frac{z^2t^2}{4}\right) \Bigg] \Bigg\} \ , \end{aligned} \tag{32}$$

where $A = a + r$.

4.1. Singular point is at the origin

In case of $z = 0$ (32) becomes:

$$f_R(t) = \frac{t^{r-1}}{\Gamma(r)} \frac{k(R,a)\cosh \pi R}{\cosh \pi R + \cos(\pi a + \pi r)} \ . \tag{33}$$

As it is follows from (33) the regularized solution rapidly tends to the exact one for all $t > 0$ as $R \to \infty$. If coefficient $k(R,a)$ in (5)is repaced with

$$k(R,a) = \frac{\cosh \pi R + \cos \pi(a+r)}{\cosh \pi R}, \tag{34}$$

then regularized solution would not depend on regularization parameter R and $f_R(t) = f(t)$.

Using the symmetry property, integral (2) can be written in the following form:

$$f_R(t) = \frac{1}{t} \int_0^\infty F\left(\frac{u}{t}\right) \Pi(R,u)\, \mathrm{d}u \ . \tag{35}$$

Then for the Laplace transform $F(p) = 1/p^r$ we get:

$$f_R(t) = t^{r-1} \int_0^\infty \frac{\Pi(R,u)}{u^r} \mathrm{d}u \ . \tag{36}$$

If we take coefficient $k(r,a)$ in form (34), then $f_R(t) = t^{r-1}/\Gamma(r)$. Therefore the following integral

$$\int_0^\infty \frac{\Pi(R,u)}{u^r} \mathrm{d}u = \frac{1}{\Gamma(r)} \tag{37}$$

does not depend on parameter R. Equality (37) is a good testing criteria for an algorithm that evaluates $f_R(t)$ using (2).

4.2. Singular point is in left half-plane

In case of $\mathrm{Re}\, z > 0$ the first term of (32) tends rapidly to the exact solution $f(t)$ as $R \to \infty$. The second term equals to zero if $a + r$ is half-integer. In general the second term is a small value only at the beginning of the process and it increases with time.

The third term (in square brackets) of (32) is more complicated. Let z be $z = \rho \exp(i\varphi)$, $-\pi/2 < \varphi < \pi/2$. Then the third term from (32) can be written as:

$$\begin{aligned} \mid E_3 \mid \sim (t\rho)^{2-a-r} \Big| &\Gamma(a+r-2+iR)(t\rho)^{-iR} \mathrm{e}^{|\varphi| R} \exp\left(i\varphi(2-a)\right) \times \\ &\times {}_1F_2\left(1; \frac{4-a-r-iR}{2}, \frac{3-a-r-iR}{2}, \frac{\rho^2 t^2 \exp(2i\varphi)}{4}\right)\Big| \ . \end{aligned} \tag{38}$$

Because of factor $(t\rho)^{-iR}$ presence, the E_3 value is an alternating quantity. It is obvious that E_3 strongly depends on angle φ. It follows from standard formulas [13], that

$$\mid \Gamma(a+r-2+iR) \mid \sim R^{a+r-5/2} \sinh^{-1/2}(\pi R) \ , \tag{39}$$

and, therefore,

$$\mid \Gamma(a+r-2+iR) \mid \exp(\varphi R) \sim R^{a+r-5/2} \exp(\varphi R - \pi R/2) \ . \tag{40}$$

Equation (40) shows that when $R \to \infty$ and $\varphi = 0$ the decreasing rate of $\mid E_3 \mid$ is highest, and it is getting slower as $|\varphi| \to \pi/2$.

Obviously the quantity $\mid E_3 \mid$ also depends on values of ρ and t. When $t\rho \ll R$ we have the following estimation:

$$\left| {}_1F_2\left(1; \frac{4-a-r-iR}{2}, \frac{3-a-r-iR}{2}; \frac{z^2t^2}{4}\right)\right| \sim 1 \ . \tag{41}$$

The factor $(\rho t)^{2-a-r}$ vanishes as $t \to 0$ if $a+r < 2$. That is we can calculate an inverse Laplace transform with small relative error for small instances of time. If condition $a + r < 2$ is not satisfied, then relative error increases as $t \to 0$ or $\rho \to 0$.

In case of $t\rho \gg R$ we can obtain asymptotic value for generalized hypergeometric function, which results in $f_R(t) \to 0$ as $t \to \infty$. That is the sum of all terms of (32) tends to zero. In considered case $f(t) \sim t^{r-1} \exp(-zt)$, $\operatorname{Re} z > 0$, so $\lim_{t\to\infty} f(t) = 0$. Thus, we can calculate the inverse Laplace transform with small absolute error as $t \to \infty$.

In applications for any given image function one should calculate the regularized solution $f_R(t)$ by calculating integral (14) with an appropriate value of parameter R which depends on inaccuracy of initial data. In case of noisy data, when a Laplace transform is known with three decimal digits, the optimal value of parameter R is approximately 3. The optimal value of R is not less than 10 when a Laplace transform can be calculated with double precision.

Figures 1, 2 show graphs of real part of absolute error δ when $a + r = 1/2$, $\rho = 1$, and $0 < t < 100$. As it is seen from graphs the original function can be found with small absolute error for all instances of time. The absolute error value strongly depends on angle φ.

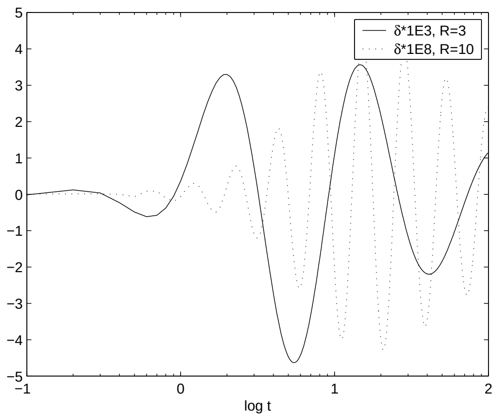

Figure 1. Real part of absolute errors for $\varphi = 0$.

4.3. Singular point is on the imaginary axis

In case when Laplace transform has a singular point on the imaginary axis error analysis is similar to the previous case. Indeed, in this case $\operatorname{Re} z = 0$ and $\varphi = \pm\pi/2$, then instead of (40) we will have:

$$\left|\Gamma(a + r - 2 - iR) \exp\frac{\pi R}{2}\right| \sim R^{a+r-5/2} \ . \tag{42}$$

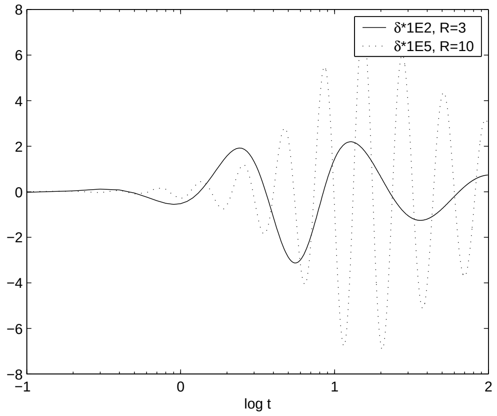

Figure 2. Real part of absolute errors for $\varphi = \pi/4$.

That is the third term in (32) decreases not slower than $R^{-1/2}$ for any valid value $0 < a+r < 2$. Thus, we have the slowest convergence to the exact solution as $R \to \infty$ when Laplace transform singularities are on imaginary axis.

In case when $t \to 0$, the value of E_3 tends to zero because $a + r < 2$. Therefore we can find the inverse Laplace transform with small relative error at the beginning of the process. If $t \to \infty$ then $f_R(t) \to 0$ as it was in the previous case. Then, if $r < 1$ we have that $f(t) \to 0$ as $t \to \infty$. That is final values of exact original function and $f_R(t)$ are equal. However, if $r \geq 1$ then $\lim_{t\to\infty} f(t)$ does not exist and the values of the original function can not be found as $t \to \infty$.

Graph in figure 3 shows real part of absolute errors when $\varphi = \pi/2$, $a + r = 1/2$, $r = 1$.

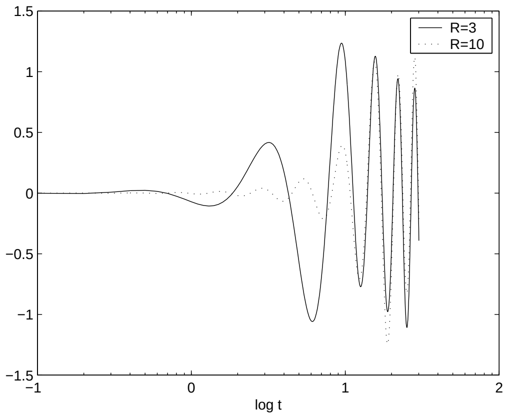

Figure 3. Real part of absolute errors for $\varphi = \pi/2$, r=1.

4.4. Conclusion

It follows from the analysis above that, in case when conditions (6), (7) are satisfied, we have that:

- if $lim_{t\to\infty} f(t)$ exists or $f(t)$ has a power asymptotic function then $f(t)$ can be approximately determined for any $t > 0$;

- if $lim_{t\to\infty} f(t)$ does not exist then $f(t)$ can be approximately determined only in some neighborhood of $t = 0$;

- the rate of convergence decreases as function $f(t)$ becomes less and less monotonous:

$$t^{r-1} \to t^{r-1}\mathrm{e}^{-\alpha t} \to t^{r-1}\mathrm{e}^{-\alpha t}\sin\omega t \to t^{r-1}\sin\omega t \tag{43}$$

For the last function in (43) we should expect to obtain acceptable results from noisy data only in some neighborhood of $t = 0$, especially in case of $r \geq 1$.

Provided error analysis reveals well known restrictions of determining values of inverse Laplace transform as $t \to \infty$ with the help of final-value or asymptotical expansion theorems of operational calculus [5].

The fact that the rate of convergence strongly depends on location of the Laplace transform singular points was first pointed out by Orurk [14].

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