ESTIMATION OF INITIAL CONDITIONS AND INVERSE PROBLEM SOLUTION FOR A DRYING SYSTEM IN A POROUS MEDIUM

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ABSTRACT

A numerical marching scheme is introduced for the recovery of the solutions, gradient distributions and initial conditions in a one dimensional Luikov's drying system in a porous medium with space dependent coefficients. In this problem, only Cauchy noisy data at the active boundary is given and no information about the amount and/or character of the noise in the data is assumed. The error analysis for the algorithm is discussed and numerical examples of interest are presented**.**

1. INTRODUCTION

Thermal drying involves the vaporization of moisture within a product by heat and the evaporation of moisture from the medium and has important applications in many different fields, including food and environmental engineering. A theoretical model for simultaneous heat and mass transfer was developed by Luikov [2].

In [1], the authors discuss a Luikov system of the form:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \eta \end{pmatrix} \begin{pmatrix} u_{xx} \\ v_{xx} \end{pmatrix}, \quad 0 < x < 1, \quad t > 0,$$

with boundary conditions

$$\begin{aligned} u_x(0,t) &= -Q, \\ v_x(0,t) &= -PnQ, \\ u_x(1,t) &= -Bi_q \quad u(1,t) + (1 - Eo)KoLu \\ &\quad \times Bi_m(v(1,t) - 1) + Bi_q V(t), \\ v_x(1,t) &= Bi_m(1 - v(1,t)) - Pn \quad u_x(1,t), \end{aligned}$$

where $V(t)$ is a transient function associated with the dry air flow, and initial conditions

$$\begin{aligned} u(x,0) &= u_0, \\ v(x,0) &= v_0. \end{aligned}$$

The constant coefficients α, β, γ and η are defined as

$$\begin{aligned} \alpha &= 1 + E_o K_o Lu Pn, \\ \beta &= -E_o K_o Lu, \\ \gamma &= -Lu Pn, \\ \eta &= Lu. \end{aligned}$$

The terms $Lu = \frac{a_m}{a}, Pn, K_o, Bi_q, Bi_m,$ and refer to the Luikov number, Possnov number, Kossovitch number, heat Biot, mass Biot, and heat flux flux respectively. The coefficients a and a_m represent the thermal diffusivity and the moisture diffusivity of the porous medium. Deterministic, stochastic, and hybrid solutions were introduced in [3] and [6] for estimation of parameters in the above problem.

In this paper we consider nonhomogeneous thermal and moisture diffusivities of the porous medium so that the Luikov number and all the coefficients, $\alpha, \beta, \gamma,$ and η of the model, are space dependent functions. We will introduce a stable numerical marching scheme based on discrete mollification for the recovery of $u(x,t)$, $v(x,t)$, $u_x(x,t)$, $v_x(x,t)$, $u(x,0)$ and $v(x,0)$ throughout the domain $[0,1] \times [0,1]$ in the (x,t) plane satisfying

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} \alpha(x) & \beta(x) \\ \gamma(x) & \eta(x) \end{pmatrix} \begin{pmatrix} u_{xx} \\ v_{xx} \end{pmatrix}, \qquad (*)$$

$$0 < x < 1, \quad 0 < t < 1,$$

with boundary conditions

$$u(0,t) = g_1(t),$$

$$v(0,t) = g_2(t),$$

$$u_x(0,t) = g_3(t),$$

$$v_x(0,t) = g_4(t).$$

Note that the functions g_1, g_2, g_3 and g_4 are only known approximately.

This problem is an inverse Cauchy problem involving a parabolic system. The implementation of these algorithms do not require any information about the amount and/or characteristics of the noise in the data since the mollification parameters are chosen automatically at each step using the Generalized Cross Validation (GCV) method. For general references to the GCV method see [7].

The paper is organized as follows: discrete mollification and numerical differentiation results are summarized in Section 2. In Section 3, the numerical space marching algorithm is specified. Stability and error estimates are also presented in this section. Section 4 contains numerical examples of interest.

2. MOLLIFICATION

A detailed description of the regularization procedure of Mollification and its applications can be found in [4].

2.1 Discrete Mollification

Let $I = [0,1]$ and $K = \{x_i : i = 1,2,...,N\} \subset I$ satisfying $0 \le x_1 < x_2 < ... < x_N \le 1$. Set $s_0 = 0$, $s_N = 1$, and $s_i = \frac{1}{2}(x_{i+1} + x_i)$ for $i = 1,2,...,N-1$. Suppose that $G = \{g_i\}_{i=1}^N$ is a discrete function defined on K, then the δ mollification of G is defined as a convolution with the Gaussian kernel

$$\rho_\delta(t) = \begin{cases} A_p \delta^{-1} \exp\left(-\frac{t^2}{\delta^2}\right), & t \in I_\delta, \\ 0, & t \notin I_\delta, \end{cases}$$

where $I_\delta = [-p\delta, p\delta]$, $\delta > 0$, $p > 0$, and $A_p = \left(\int_{-p}^{p} \exp(-s^2) ds\right)^{-1}$. That is, for every $x \in I_\delta, J_\delta G(x) = \sum_{i=1}^{N} \left(\int_{s_{i-1}}^{s_i} \rho_\delta(x-s) ds\right) g_i$.

2.2 Numerical Differentiation

The centered finite difference operator, $\mathbf{D}_0 f(x) = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}$, is defined on $\tilde{I}_\delta = [p\delta + \Delta x, 1 - p\delta - \Delta x]$.

Let $G^\epsilon = \{g_i + \epsilon_i : |\epsilon_i| \le \epsilon, i = 1,2,...,N\}$ be a perturbed discrete version of a function g, where ϵ is the maximum noise level. The following lemma, establishes the numerical convergence of centered difference discrete mollified differentiation for a fixed δ.

Lemma 2.1
If g is uniformly Lipschitz on I and the discrete functions G and G^ϵ satisfy $\|G - G^\epsilon\|_{\infty,K} \le \epsilon$, then there exist constants C, independent of δ, and C_δ, such that

$$\| \mathbf{D}_0(J_\delta G^\epsilon) - \frac{\partial}{\partial x} J_\delta g \|_{\infty,\tilde{I}_\delta} \le C(\epsilon + \Delta x).$$

We define the discrete mollified centered difference $\mathbf{D}_0^\delta(G) = \mathbf{D}_0(J_\delta G)|_{\tilde{I}_\delta \cap K}$, by restricting $\mathbf{D}_0(J_\delta G)$ to the grid points of $\tilde{I}_\delta \cap K$. The next theorem establishes a useful upper bound for the operator $\mathbf{D}_0^\delta$.

Theorem 2.2
There exists a constant C, independent of δ, such that

$$\| \mathbf{D}_0^\delta G \|_{\infty, K\cap\tilde{I}_\delta} \le \frac{C}{\delta} \| G \|_{\infty,K}.$$

For the proof of these statements see [4].

3. THE IDENTIFICATION PROBLEM

The problem consists on the identification of the vapor diffusion, $u(x,t)$, initial vapor distribution, $u(x,0)$, vapor flux $u_x(x,t)$, moisture diffusion $v(x,t)$, initial moisture distribution, $v(x,0)$ and moisture flux, $v_x(x,t)$, for all (x,t) throughout the domain $[0,1]\times[0,1]$ satisfying system $(*)$.

The available data $g_1^{\epsilon}, g_2^{\epsilon}, g_3^{\epsilon}$, and g_4^{ϵ} are discrete noisy functions with maximum noise level ϵ. We define $A(x) = \begin{pmatrix} \alpha(x) & \beta(x) \\ \gamma(x) & \eta(x) \end{pmatrix}$ and assume that $|\ det(A(x))\ | \geq d > 0$ for all $x \in [0,1]$.

We begin by stabilizing the problem using mollification. In this regularization process, a δ - mollification is performed on each of the available data functions, g_1^{ϵ}, g_2^{ϵ}, g_3^{ϵ}, and g_4^{ϵ}. Note that δ - mollifications of g_1^{ϵ}, g_2^{ϵ}, g_3^{ϵ}, and g_4^{ϵ} are taken with respect to t using δ_u^0, δ_v^0, δ_{ux}^0 and δ_{vx}^0 respectively.

The numerical marching scheme, together with the mollification method, is described next with $\tilde{u}(x,t)$ and $\tilde{v}(x,t)$ denoting the regularized functions.

3.1 Numerical Marching Scheme

Let N_x and N_t be positive integers, $\Delta x = h = \frac{1}{N_x}$, $\Delta t = k = \frac{1}{N_t}$, $x_i = ih$, $i = 0,1,\ldots,N_x$, and $t_n = nk$, $n = 0,1,\ldots,N_t$.

We introduce the following discrete functions

$R_u^{i,n}$: discrete approximation to $\tilde{u}(ih,nk)$,

$R_v^{i,n}$: discrete approximation to $\tilde{v}(ih,nk)$,

$Q_u^{i,n}$: discrete approximation to $\tilde{u}_x(ih,nk)$,

$Q_v^{i,n}$: discrete approximation to $\tilde{v}_x(ih,nk)$,

$W_u^{i,n}$: discrete approximation to $\tilde{u}_t(ih,nk)$,

$W_v^{i,n}$: discrete approximation to $\tilde{v}_t(ih,nk)$,

$S_u^{i,n}$: discrete approximation to $\tilde{u}_{xt}(ih,nk)$,

$S_v^{i,n}$: discrete approximation to $\tilde{v}_{xt}(ih,nk)$.

The space marching algorithm is defined as follows:

1. Select δ_u^0, δ_v^0, δ_{ux}^0, and δ_{vx}^0.
2. Perform mollification of g_1^{ϵ}, g_2^{ϵ}, and g_4^{ϵ}.

Set:

- o $R_u^{0,n} = J_{\delta_u^0} g_1^{\epsilon}(nk)$,
  $R_v^{0,n} = J_{\delta_v^0} g_2^{\epsilon}(nk)$.
- o $Q_u^{0,n} = J_{\delta_{ux}^0} g_3^{\epsilon}(nk)$,
  $Q_v^{0,n} = J_{\delta_{vx}^0} g_4^{\epsilon}(nk)$.

3. Perform mollified differentiation in time of $J_{\delta_u^0} g_1^{\epsilon}(nk), J_{\delta_v^0} g_2^{\epsilon}(nk), J_{\delta_{ux}^0} g_3^{\epsilon}(nk), J_{\delta_{vx}^0} g_4^{\epsilon}(nk)$.

Set:

- o $W_u^{0,n} = \mathbf{D}_t(J_{\delta_u^0} g_1^{\epsilon}(nk))$ and
  $W_v^{0,n} = \mathbf{D}_t(J_{\delta_v^0} g_2^{\epsilon}(nk))$.
- o $S_u^{0,n} = \mathbf{D}_t(J_{\delta_{ux}^0} g_3^{\epsilon}(nk))$ and
  $S_v^{0,n} = \mathbf{D}_t(J_{\delta_{vx}^0} g_4^{\epsilon}(nk))$.

The numerical marching scheme in space is defined in step 4.

4. Initialize $i = 0$. Do while $i \leq N_x - 1$.
   (a) $R_u^{i+1,n} = R_u^{i,n} + h\,Q_u^{i,n}$ and
   $R_v^{i+1,n} = R_v^{i,n} + h\,Q_v^{i,n}$.
   (b) $Q_u^{i+1,n} = Q_u^{i,n} + \frac{h}{det(A(ih))}$
   $\times(-\gamma(ih)W_u^{i,n} + \alpha(ih)W_v^{i,n})$.
   (c) $Q_v^{i+1,n} = Q_v^{i,n} + \frac{h}{det(A(ih))}$
   $\times(-\gamma(ih)W_u^{i,n} + \alpha(ih)W_v^{i,n})$
   (d) Select $\delta_u^{i+1}, \delta_v^{i+1}, \delta_{ux}^{i+1}, \delta_{vx}^{i+1}$.
   (e) Perform mollified differentiation in time of $R_u^{i+1,n}, R_v^{i+1,n}, Q_u^{i+1,n},\ Q_v^{i+1,n}$.
   Set :
   - o. $W_u^{i+1,n} = \mathbf{D}_t(J_{\delta_u^{i+1}} R_u^{i+1,n})$ and
     $W_v^{i+1,n} = \mathbf{D}_t(J_{\delta_v^{i+1}} R_v^{i+1,n})$.
   - o $S_u^{i+1,n} = \mathbf{D}_t(J_{\delta_{ux}^{i+1}} Q_u^{i+1,n})$ and
     $S_v^{i+1,n} = \mathbf{D}_t(J_{\delta_{vx}^{i+1}} S_v^{i+1,n})$.

   (f) Set $i = i + 1$.

Remark: The discrete approximations $\tilde{u}(x,0)$ and $\tilde{v}(x,0)$ are given by $R_u^{i,0}$ and $R_v^{i,0}$, respectively.

For a proof of the statements in the next two subsections see [1].

3.2 Stability Analysis

Denote $|Y^i| = \max_n |Y^{i,n}|$ and $\|Y\|_\infty = \max_i |Y^i|$. Theorem 2.1 and Theorem 2.2 establish stability and formal convergence, respectively, of the marching scheme presented above.

Theorem 3.1
There exists a constant C_0 such that

$$\max\{|R_u^L|,|R_v^L|,|Q_u^L|,|Q_v^L|\} \leq$$
$$\exp(C_0)\max\{|R_u^0|,|R_v^0|,|Q_u^0|,|Q_v^0|\}.$$

3.3 Error Estimates

Denoting the error between the calculated discrete functions $R_u^{i,n}, Q_u^{i,n}$ and the restriction to the grid of the mollified exact functions $\tilde{u}(ih,nk), \tilde{u}_x(ih,nk)$ by $\Delta R_u^{i,n} = R_u^{i,n} - \tilde{u}(ih,nk)$ and $\Delta Q_u^{i,n} = Q_u^{i,n} - \tilde{u}_x(ih,nk)$, proceeding similarly with the discrete functions related to $v(x,t)$, we define $\Delta_i = \max\{|\Delta R_u^i|, |\Delta R_v^i|, |\Delta Q_u^i|, |\Delta Q_v^i|\}$.

Theorem 3.2
There exists a constant C_0 such that

$$\Delta_L \leq \exp(C_0)(\Delta_0 + \epsilon + k).$$

4. NUMERICAL IMPLEMENTATION

In this section the numerical results of an example of interest is presented. To obtain the required data functions $u(0,t)$ and $v(0,t)$ for the inverse problem, it is necessary first to solve the direct problem. We set the following dimensionless values for the parameters in Luikov's model:

$$\begin{aligned} Lu &= 0.8(1+x), \\ Pn &= 0.32, \\ Ko &= 65, \\ Eo &= 0.02, \\ Bi_q &= 1.7, \\ Bi_m &= 3.0, \\ Q &= 2.5. \end{aligned}$$

Thus, the system of partial differential equations becomes

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = [(1+x)\begin{pmatrix} 0.7488 & -1.04 \\ -0.256 & 0.8 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}]\begin{pmatrix} u_{xx} \\ v_{xx} \end{pmatrix},$$

$$0 < x < 1, \quad 0 < t \leq 1,$$

with boundary conditions

$$\begin{aligned} u_x(0,t) &= -2.5, \\ v_x(0,t) &= -0.8, \\ u_x(1,t) &= -1.7 \quad u(1,t) + \\ &\quad 152.88(1+x)(v(1,t)-1) \\ &\quad +1.7V(t), \\ v_x(1,t) &= 3(1-v(1,t)) - 0.32 \quad u_x(1,t), \end{aligned}$$

and initial conditions

$$\begin{aligned} u(x,0) &= 2.5\ x\ (x\ g(0)-1), \\ v(x,0) &= 1.5 + 0.8\ x\ (x-1). \end{aligned}$$

The functions

$$V(t) = (u(1,0) + \frac{v_x(1,0)(1-E_0)K_oLu}{Bi_q}) \times (-9 + 10e^{t^2})$$

and $g(t) = 3.1 - t$, are chosen to satisfy the required compatibility conditions at $t = 0$ to avoid potential space located patches in the solution for positive times that will render the solution of the inverse problem impossible. See [5].

The numerical solution of the direct problem is computed by the method of lines and the discrete perturbed data functions for the inverse problem are generated by adding random errors to the "exact" computed solutions of the direct problem $g_1 = u(0,t)$ and $g_2 = v(0,t)$ as well as the exact flux functions $g_3 = u_x(0,t) = -2.5$ and $g_4 = v_x(0,t) = -0.8$. That is, $g_i^\epsilon = g_i + \epsilon_i$ where the ϵ_i's are Gaussian random variables with $|\epsilon_i| \leq \in, \quad i = 1,2,3,4.$

The relative weighted l^2 error for u is calculated as

$$\frac{[\frac{1}{(M+1)}\sum_{i=0}^{M} | R_u^i - u(ih) |^2]^{\frac{1}{2}}}{[\frac{1}{(M+1)}\sum_{i=0}^{M} | u(ih) |^2]^{\frac{1}{2}}}.$$

The relative l^2 errors for u_0, u_x, v_0, v and v_x are computed in a similar fashion.

Example

We wish to approximately identify the functions $u(x,t)$, $v(x,t)$, $u(x,0)$, $v(x,0)$, $u_x(x,t)$, and $v_x(x,t)$ satisfying

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = [(1+x)\begin{pmatrix} 0.7488 & -1.04 \\ -0.256 & 0.8 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}]\begin{pmatrix} u_{xx} \\ v_{xx} \end{pmatrix},$$

$$0 < x < 1, \;\; 0 < t \leq 1,$$

and the boundary conditions

$$u(0,t) = g_1^\epsilon(t),$$

$$v(0,t) = g_2^\epsilon(t),$$

$$u_x(0,t) = g_3^\epsilon(t),$$

$$v_x(0,t) = g_4^\epsilon(t).$$

Relative l^2 errors for u and v are reported in Table 1 as a function of ϵ and as a function of Δt in Table 2. Both these results and those shown in Figures 1 through 6 emphasize the stability and consistency of the marching scheme. For Table 1 and Figures 1 through 6, $N_x = 100$ and $N_t = 128$. In Table 2 and Figures 1 through 6, $\epsilon = 0.01$.

Acknowledgement

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ϵ	$u(x,t)$	$v(x,t)$
0.001	0.00745	0.04766
0.005	0.00792	0.04435
0.01	0.00953	0.12241

Table 1

Δt	$u(x,t)$	$v(x,t)$
1/64	0.00959	0.12421
1/128	0.00953	0.12241
1/256	0.00452	0.01227

Table 2

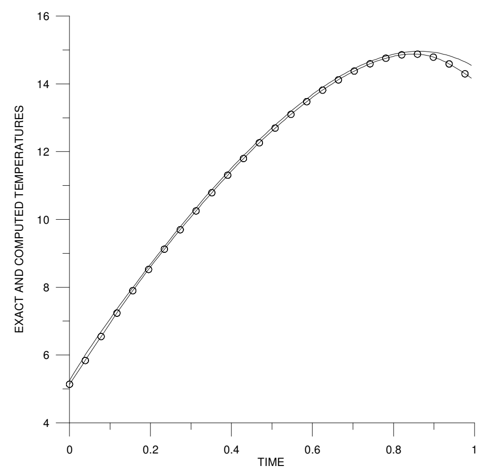

Fig. 1. Exact and computed temperatures at $x = 1$

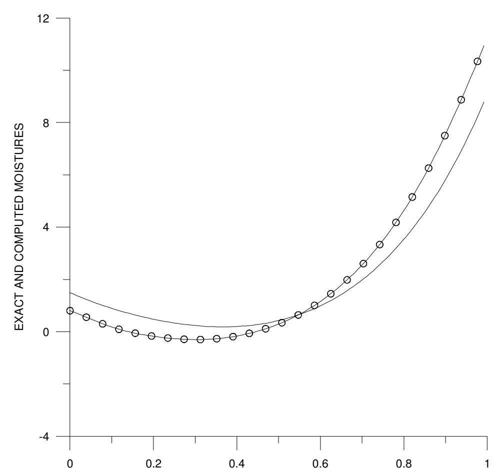

Fig. 2. Exact and computed moistures at $x = 1$

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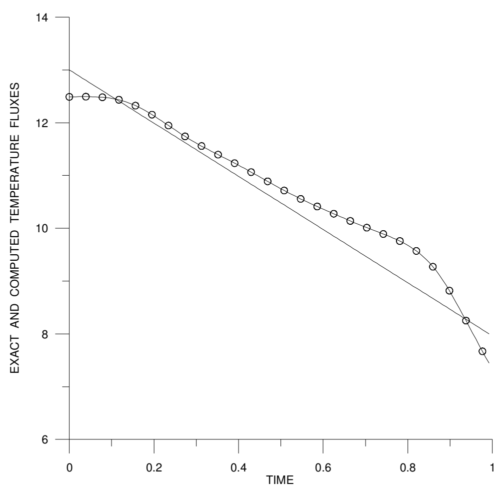

Fig. 3. Exact and computed heat fluxes at $x = 1$

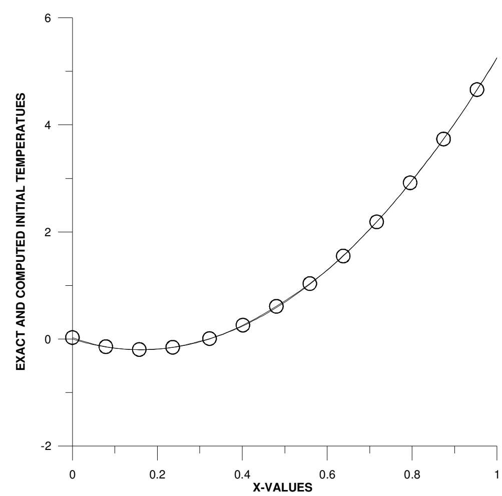

Fig. 5. Exact and computed temperatures at $t = 0$

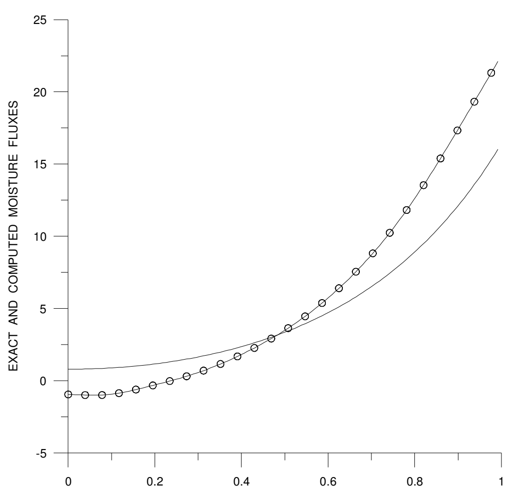

Fig. 4. Exact and computed moisture fluxes at $x = 1$

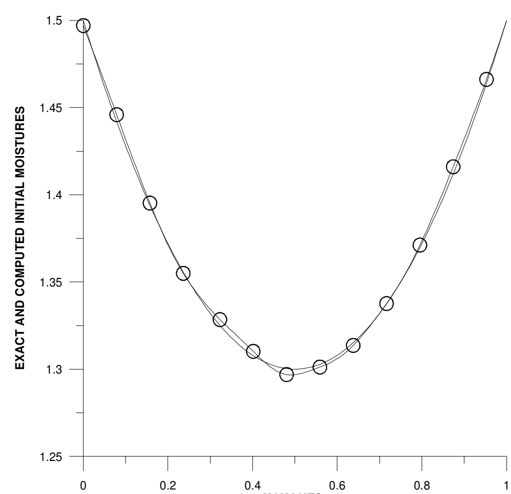

Fig. 6. Exact and computed moistures at $t = 0$