Inverse obstacle back-scattering problem with modulus data

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1. Inverse sound-soft obstacle scattering problems

$$\begin{cases} u = u^i + u^s, \ \ u^i = e^{ikx\cdot d} & \\ \Delta u + k^2 u = 0, & \text{in } \mathbb{R}^2 \setminus \overline{\Omega} \\ u = 0, & \text{on } \partial\Omega \\ \lim_{|x|\to\infty} \sqrt{|x|}(\frac{\partial u^s}{\partial |x|} - iku^s) = 0 & \end{cases}$$

$$u_\infty(\widehat{x}, d, k) = \lim_{|x|\to\infty} u^s(x,d)\sqrt{|x|}e^{-ik|x|}$$

- Inverse obstacle scattering problem:

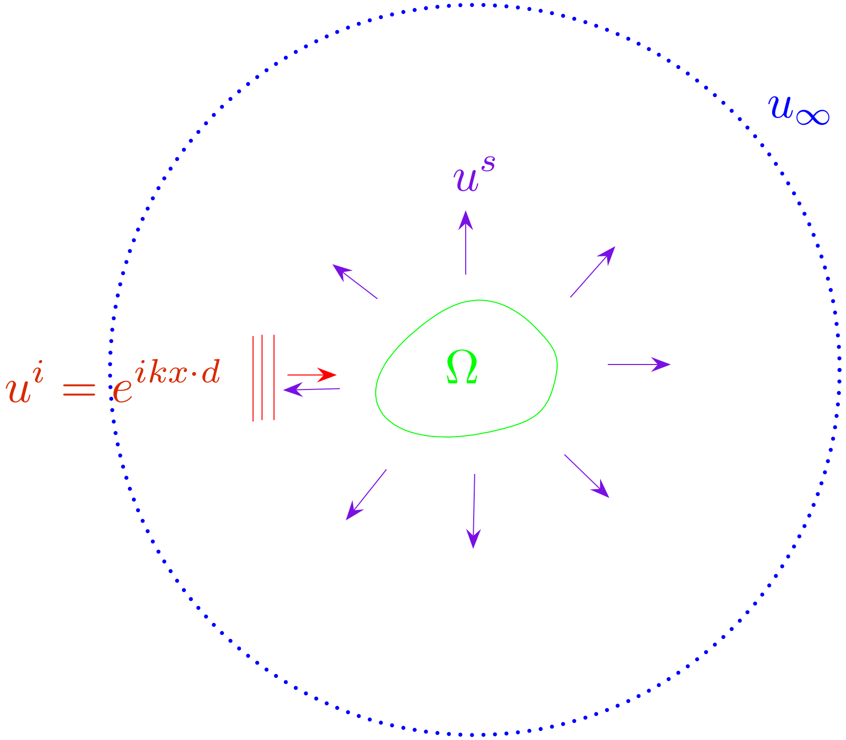
Recover $\partial\Omega$ from $u_\infty(\widehat{x}, d, k)$ in a certain set $E \subset S^1 \times S^1 \times \mathbb{R}_+$,

i.e. solve the nonlinear and ill posed equation

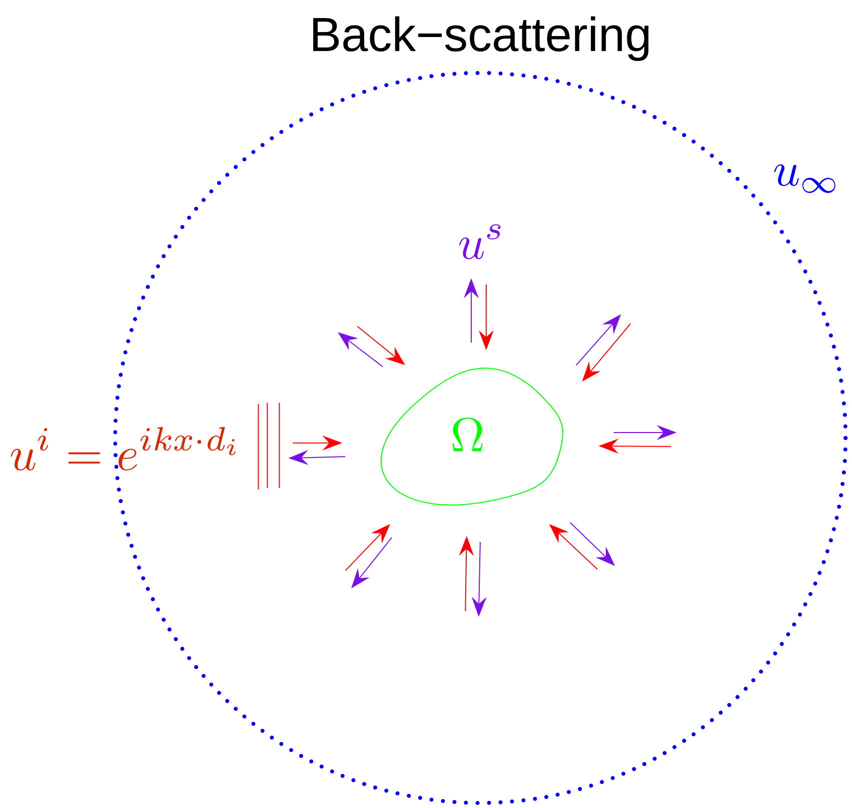
$$F(\partial\Omega) = u_\infty$$

for the unknown boundary $\partial\Omega$

Fixed incident direction



$$E = S^1 \times \{d = d_0\} \times \{k = k_0\}$$



$$E = \{(\widehat{x}, d) : \widehat{x} = -d\} \times \{k = k_0\}$$

2. Previous results

- $E = S^1 \times S^1 \times \{k = k_0\}$

  ▶ Uniqueness: Schiffer[8] (1967)

- $E = S^1 \times \{d = d_0\} \times \{k_i : 1 \leq i \leq n\}$

  ▶ Uniqueness: Colton & Sleeman[2] (1983)

  ▶ Linearization of F: Kirsch[4] (1993), Potthast[10] (1994)

  ▶ Numerical implementation: Kirsch[5] (1993), Kress & Rundell[6] (1994)

- $E = \{(\widehat{x}, d) : \widehat{x} = -d\} \times K$

  ▶ Uniqueness: Majda[9](1974) with assuming that Ω is convex and K has a limit point.

  ▶ Linearization

  ▶ Uniqueness of linearized problem: Hähner and Kress[3] (2000) showed uniqueness of linearized problem with $K = \{k : 0 < k < 1\}$

3. Modulus data

- Overdetermined problem

data($u_\infty(\widehat{x}, d_0, k_0)$): complex valued function
$\longrightarrow$ solution ($\partial\Omega(r(t))$): real valued function

- translation invariance

$$\begin{aligned} F(\partial\Omega) &= u_\infty(\widehat{x}, d, k) \\ F(\partial\Omega + h) &= e^{ikh\cdot(d-\widehat{x})} u_\infty(\widehat{x}, d, k) \\ |F(\partial\Omega)| &= |F(\partial\Omega + h)| \end{aligned}$$

- We don't have analogue for Schiffer[8]'s uniqueness result even with the translation invariance taken into account.

- Majda[9](1976): If Ω is a smooth strictly convex, and $\widehat{x} \neq d$, then for sufficiently large k,

$$|u_\infty(\widehat{x}, d, k) - \frac{e^{iky^+\cdot(\widehat{x}-d)}}{\sqrt{\kappa(y^+)}} \frac{< \frac{d-\widehat{x}}{|d-\widehat{x}|}, \widehat{x} >}{\sqrt{|d-\widehat{x}|}}| \leq C\frac{1}{k}$$

where $\nu(y^+) = \frac{d-\widehat{x}}{|d-\widehat{x}|}$ for the Gauss map ν, and $\kappa(y^+)$ is the curvature at y^+.

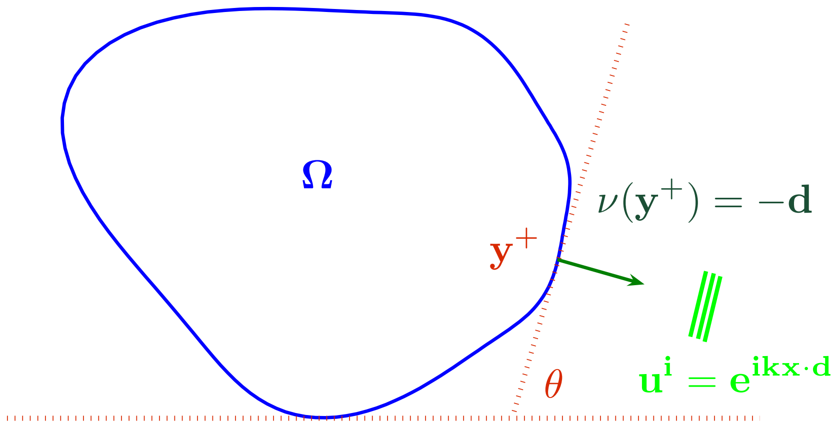
- Kress, Rundell[7] (1997)

$$\begin{aligned}
D(|F(\partial\Omega)|^2)q &= 2\,\mathrm{Re}\,\overline{F(\partial\Omega)}DF(\partial\Omega)q \\
r_{n+1} &= r_n - A_n[|F(r_n)|^2 - |u_\infty|^2] \\
\{\sin\varphi, \cos\varphi\} &\subset \ker D(|F(\partial B(0,1))|^2)
\end{aligned}$$

4. Quasi-Newton Method

$$G_k(\kappa(\theta)) = \frac{1}{2|u_\infty(-d(\theta), d(\theta), k)|^2}$$

where, $$\nu(y^+) = -d = (\cos(\theta + \frac{\pi}{2}),\ \sin(\theta + \frac{\pi}{2}))$$



Majda result or the Kirchhoff approximation and the stationary phase method give

$$\lim_{k\to\infty} G_k(\kappa(\theta)) = \kappa(\theta)$$

$$\kappa_{n+1} = \kappa_n - A_n[G_k(\kappa_n) - \frac{1}{2|u_\infty|^2}]$$

$$\begin{cases} \text{Newton method;} & A_n = DG_k(\kappa_n)^{-1} \\ \text{Kress-Rundell[7];} & A_n = \text{regularization methods involving } DG_k(1) \\ \text{Our work;} & A_n = DG_\infty(\kappa_n)^{-1} = I \end{cases}$$

$$\kappa_{n+1} = \kappa_n + \frac{1}{2|u_\infty|^2} - G_k(\kappa_n)$$

5. Recover the domain from the curvature[1]

$$v(\theta) = \frac{1}{\kappa(\theta)}$$

$$\begin{cases} x_1(\theta) = & \int_0^\theta \frac{1}{\kappa(\sigma)} \cos\sigma d\sigma \\ x_2(\theta) = & \int_0^\theta \frac{1}{\kappa(\sigma)} \sin\sigma d\sigma \end{cases}$$

6. Numerical results

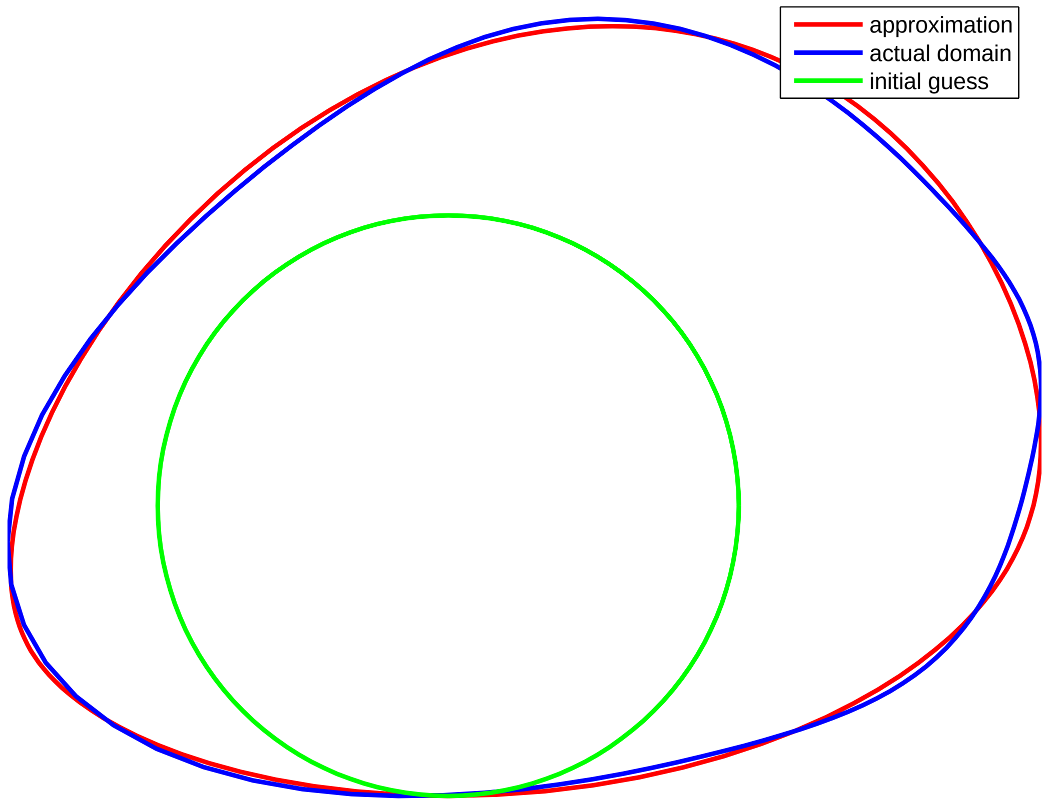


figure1: $n = 128, k = 0.5, 40$ iterations

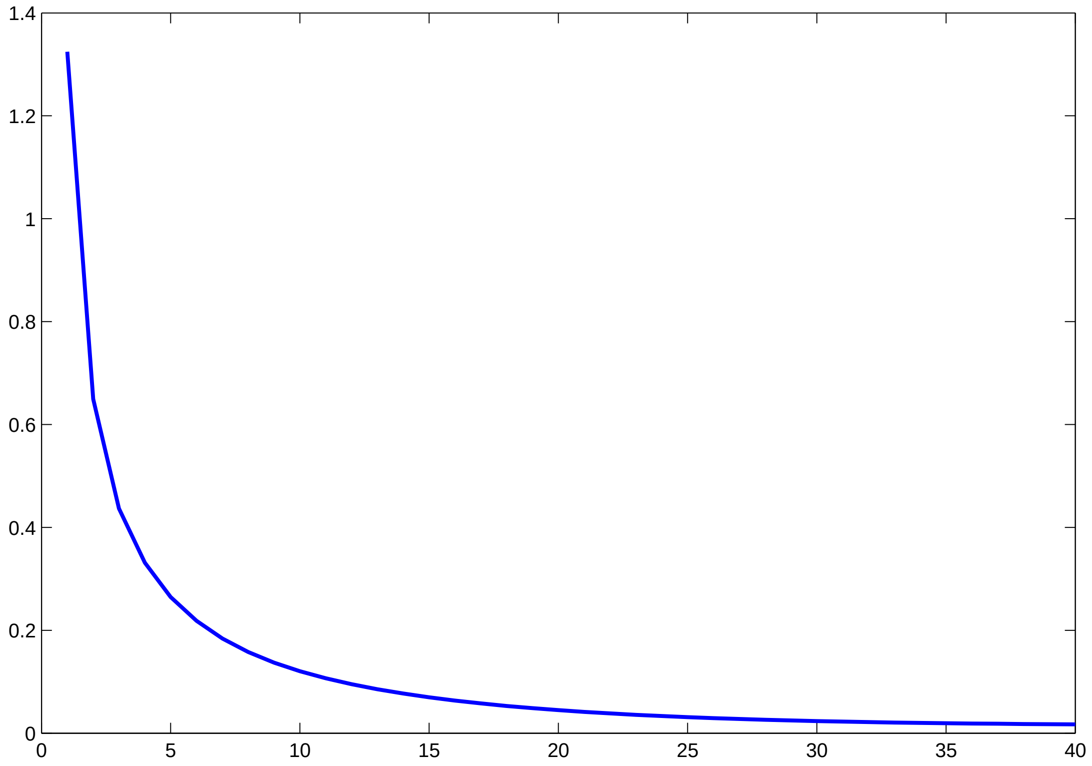


figure2: residual$= ||\frac{1}{2|u_\infty|^2} - G_k(\kappa_n)||_2$

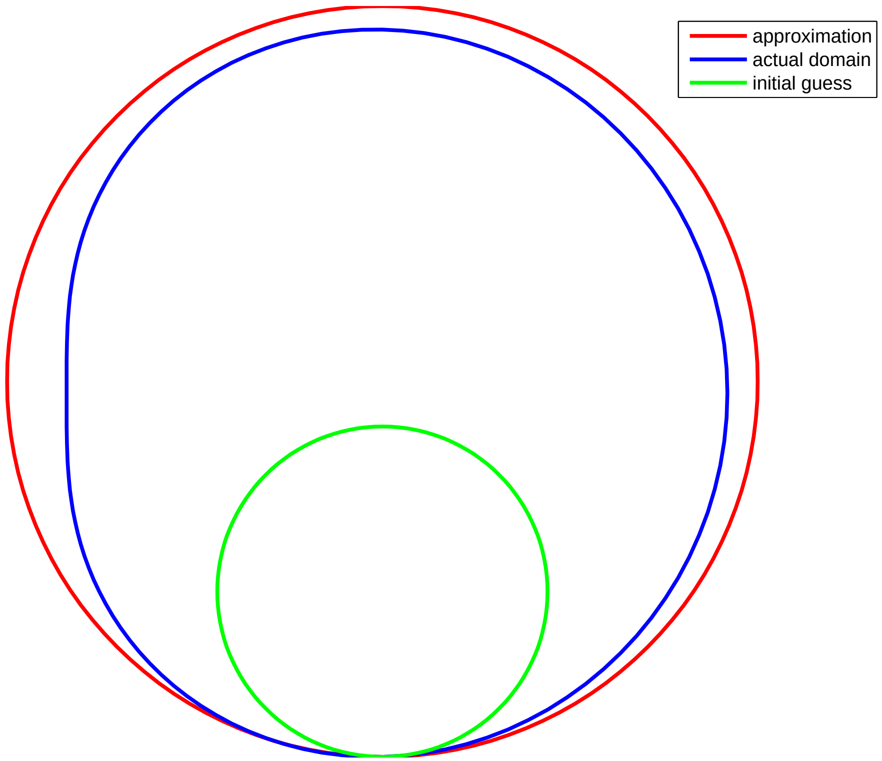


figure3: $n = 128, k = 0.001, 300$ iterations

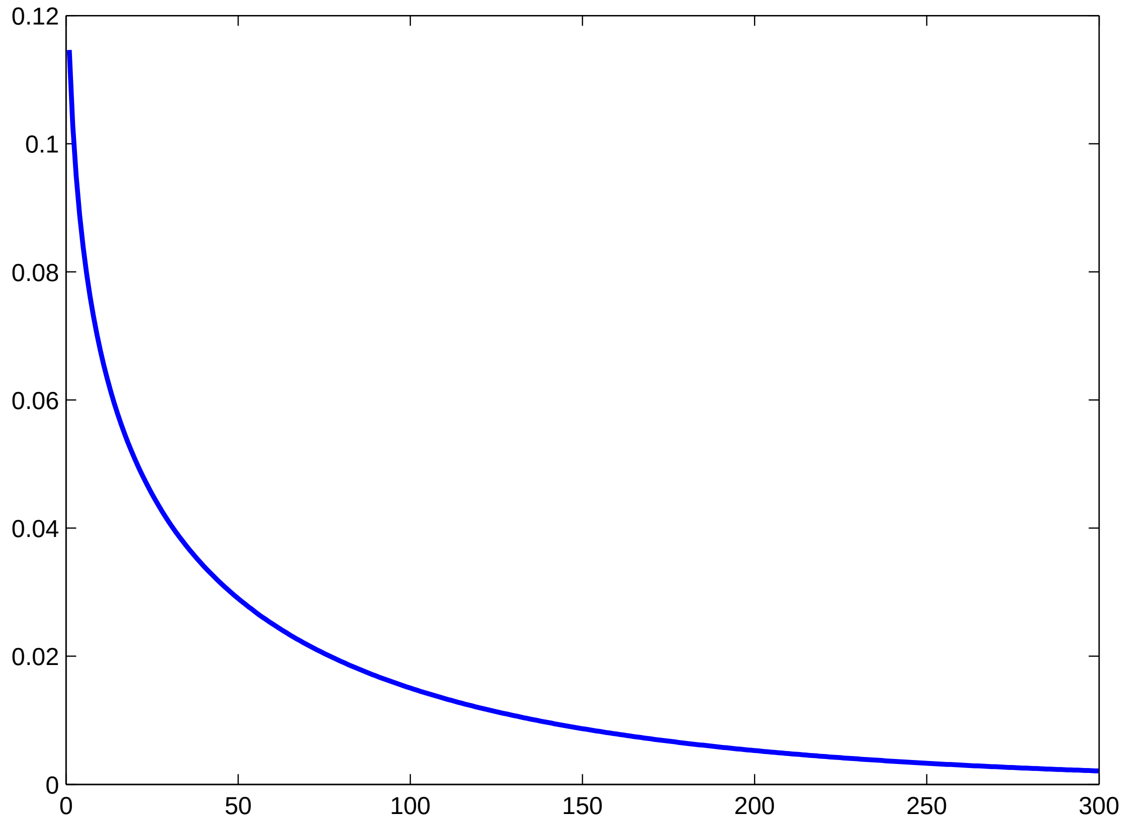


figure4: residual$= ||\frac{1}{2|u_\infty|^2} - G_k(\kappa_n)||_2$

7. Convergence and uniqueness

- Define a operator

$$\phi(\kappa) = \kappa - G_k(\kappa) + \frac{1}{2|u_\infty|^2}$$

We can have uniqueness as well as convergence of the new algorithm if ϕ is a contraction mapping on a subset A of suitable function space H.

$$||D\phi(\kappa)|| = ||I - DG_k(\kappa)|| \leq C < 1 \text{ for all } \kappa \in A \text{ and } k \in K$$

- Let $A = \{$ circle with radius $> r_0\}$, and for $k \in (k_0, \infty)$ with $k_0 \cdot r_0 > M$

$$||D\phi(\kappa)|| < 1$$

8. Current works and plans for the future

- Show the convergence of the algorithm, or determine a set A in which the mapping ϕ has the contraction property.

- Adopt the algorithm to the nonconvex body;

  We can recover the convex part with high frequency data.

- The non-zero off set problem. i.e.

$$E = \{(\widehat{x}, d) : \widehat{x} = -d + \alpha, \alpha \in \Sigma\} \times \{k = k_0\}$$

  for some set Σ

- Consider real part of u_∞ as a data instead of $|u_\infty|$

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