# Complex Modal Estimation of Wave Parameters in One-Dimensional Media

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## Premise

Traveling waves can be expressed in terms of complex modes of vibration

Example, traveling wave y(x,t):

$$y(x,t) = 2A\sin(\gamma x - \omega t) = A[\cos\omega t \sin\gamma x - \sin\omega t \cos\gamma x]$$

or

$$y(x,t) = \frac{1}{2}(z(x,t) + \bar{z}(x,t)$$

where $z(x,t) = Ae^{i\omega t}\phi(x)$ is the complex modal motion, and

$$\phi(x) = \sin\gamma x + i\cos\gamma x$$

is the complex mode shape.

Complex modes can be extracted from sampled time-history data of wave behavior.

## Goals

- Characterize modes in traveling dispersive waves
- Exhibit the energy distribution among the modes
- Use decomposed modes to estimate modal frequencies, wave numbers, and phase velocities
- Estimate the group velocity spectrum

## Complex Modal Decomposition [1]

- Measure $y(x_j, t_k)$ $\quad j = 1,\dots, m\,;\, k = 1,\dots, n$
- Convert to complex analytic signals $z_{jk} = z(x_j, t_k)$
- Build complex ensemble matrix $\underline{Z}$ such that $\underline{Z}$ has elements $z_{jk}$ (rows are sensor histories)
- Correlation Matrix $\quad \underline{R} = \frac{1}{n}\underline{Z}\underline{\bar{Z}}^T$
- Eigenvalue problem (matrix form)

$$\underline{R}\,\underline{U} = \underline{U}\,\underline{\Lambda}$$

- Modal matrix $\underline{U}$
- **Complex modal vectors $\underline{u}_i$** are columns of $\underline{U}$ called Complex Orthogonal Modes (**COMs**)
- **Squared Modal amplitudes** are eigenvalues $\underline{\Lambda}$ called Complex Orthogonal Values (**COVs**)
- Complex modal coordinate sampled histories are the rows of the complex modal ensemble given by $\quad \underline{Q} = \underline{\bar{U}}^T\underline{Z}$

## Example System: Euler-Bernoulli Beam

Undamped, infinite, thin rectangular steel beam

Young's modulus E = 200e9 N/m2

width b = 1 cm

height h = 1 mm

density $\rho$ = 7860 kg/m$^3$

I = bh$^3$/12, A = bh

According to the theory of waves [2]

$$\omega = a\gamma^2 \qquad (1)$$

where $\omega$ is the frequency, $\gamma$ is the wavenumber, and

$$a = \sqrt{\frac{EI}{\rho A}}$$

Phase velocity $\quad c = \omega/\gamma$

Group velocity $\quad c_g = \frac{d\omega}{d\gamma}$

## Simulation Example 1

Two-harmonic dispersive wave

$$y(x,t) = A_1\sin(\gamma_1 x - \omega_1 t) + A_2\sin(\gamma_2 x - \omega_2 t)$$

where $\quad A_1 = 1 \quad A_2 = 1/2 \quad$ mm

$\gamma_1 = 20 \quad \gamma_2 = 16 \quad$ rad/m

According to equation (1)

$\omega_1 = 582.4694 \quad \omega_2 = 372.7804 \quad$ rad/sec

$c_1 = 29.1235 \quad c_2 = 23.2988 \quad$ m/sec

According to equation (2), group velocity

$c_g = 52.4222 \quad$ m/sec

Added noise, uniformly distributed ±2$^{-6}$ times $y_{max}$

Estimated parameters

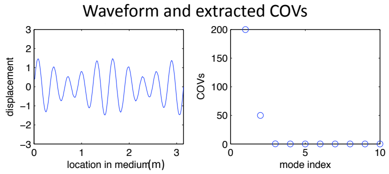
$\gamma_{1e} = 20.0000 \quad \gamma_{2e} = 16.0021 \quad$ rad/m

$\omega_{1e} = 584.9966 \quad \omega_{2e} = 375.5177 \quad$ rad/sec

$c_{1e} = 29.2498 \quad c_{2e} = 23.4668 \quad$ m/sec

$c_{ge} = 52.3803 \quad$ m/sec

### Waveform and extracted COVs




## Example 2: Disturbed Beam

Response to a Gaussian disturbance given by [2]

$$y(x,t) = \sqrt{2}f_0 b_0 s^{1/4}(t)e^{-x^2 b_0^2 s(t)}\cos[atx^2 s(t) - \phi(t)]$$

$$s(t) = \frac{1}{4(b_0^4 + a^2t^2)} \qquad \phi(t) = \tan^{-1}\frac{at}{b_0^2}$$

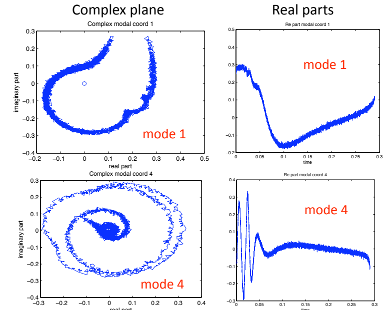
Added noise, uniformly distributed ±2$^{-6}$ times $y_{max}$

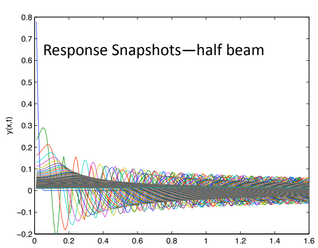
Spatial and temporal sampling limitations (sampling intervals and record lengths):

$\gamma_{min} = 3.9270 \quad \gamma_{max} = 314.15 \quad$ rad/m

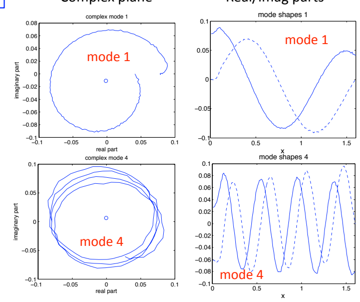
$\omega_{min} = 22.4560 \quad \omega_{max} = 375.5177 \quad$ rad/sec

### Extracted Modal Coordinates

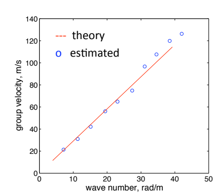


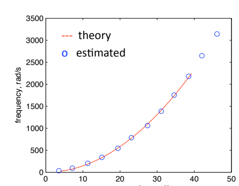

### Complex modes




Group velocity vs. wave number

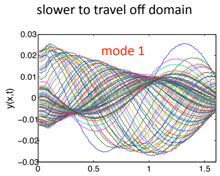



Frequency vs. wave number

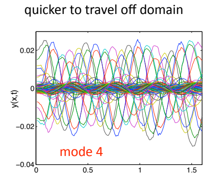



### Modal responses

First mode: longer wavelength, slower to travel off domain




Fourth mode: shorter wavelength, quicker to travel off domain




## Summary and Conclusion

Traveling waves can be described with complex modes

A complex modal decomposition can extract modal information

Modal extractions can be used to estimate wave parameters: wavelength, frequency, phase velocity, group velocity, modal energy

Decomposition is robust to noise

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## References

1. Feeny, B. F., 2008, "A complex orthogonal decomposition for wave motion analysis," *Journal of Sound and Vibration*, **310** (1-2) 77-90.
2. Graff, K. F., 1975, *Wave Motion in Elastic Solids*, Ohio State University Press, printed at the Universities Press, Belfast.